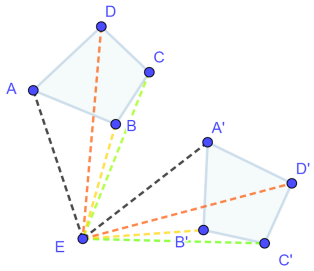
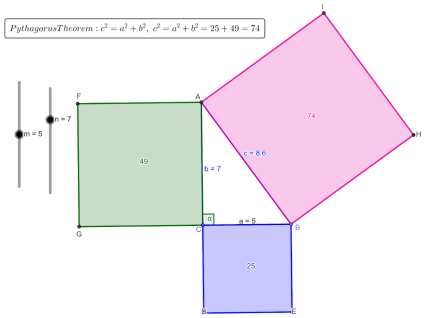
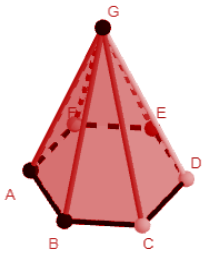
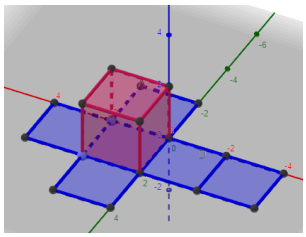
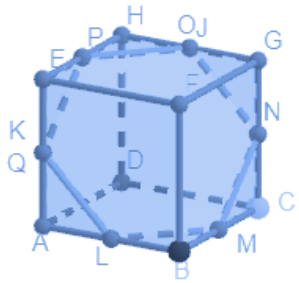
Empowering Geometry Education by Integrating it with GeoGebra

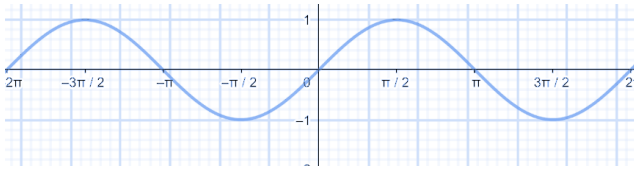












Harinder Mahajan Ph.D

Preface

In this book, we embark on a journey that combines the power of GeoGebra, with the study of High School Geometry.

GeoGebra is a free online interactive app with applications to Algebra, Geometry, Functions, Trigonometry, Statistics and Calculus. Although we will restrict its application to the study of Geometry, it will facilitate its application to other branches of Mathematics too.

GeoGebra provides a unique platform for visualization and active engagement in mathematical exploration of geometrical objects in both two and three dimensions, graphing functions, and making and testing conjectures.

A growing body of research attests to the efficacy of GeoGebra, demonstrating its capacity not only to enhance learning but also to mitigate Mathematics anxiety. Moreover, it kindles an increased interest in Mathematics, bolsters motivation, fosters greater engagement, and introduces an element of fun into the learning process.

Nevertheless, there is a need to provide a comprehensive source for integrating GeoGebra with the objectives of teaching Geometry. This will facilitate a deeper comprehension of Geometry through the avenues of visualization, active exploration, and discovery. This book serves as a valuable resource for Mathematics teachers seeking innovative and engaging methods to introduce, reinforce, and extend geometric knowledge. It is equally beneficial for homeschooling parents and self-directed learners and summer courses that may be offered to prepare students for a course in Geometry.

In this book, we will dive deep into GeoGebra's features to visualize geometric objects, explore their properties, measure their attributes and make conjectures about them, make transformations that preserve both shape and size or shape only and prove congruence and similarity of figures by transformations, and draw graphs of functions. We will delve into various branches of geometry, from Euclidean geometry and trigonometry to coordinate geometry, focusing on integrating technology seamlessly into the learning process.

It does not cover all objectives of teaching Geometry, it is only to supplement them. For more on concepts and other objectives see my presentation:
https://docs.google.com/presentation/d/1lj4-7Ci5r9B5NG9jfbnMNSqdn1rruk6VXJrvXgSzKeM/edit?usp=sharing

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1. Introduction to GeoGebra

What is GeoGebra

GeoGebra is a powerful, free online interactive application designed for teaching and learning Algebra, Geometry, Functions, Trigonometry, Statistics, and Calculus. It caters to students from elementary school to university, offering a visually appealing and engaging approach to learning mathematics.

Why use GeoGebra

Research consistently demonstrates that GeoGebra enhances visualization, retention, reasoning, and communication skills. It elevates students to higher levels of geometric thinking, thus helping in problem-solving and proofs in Euclidean Geometry. Additionally, GeoGebra has been shown to reduce mathematics anxiety, increase interest, motivation, engagement, and overall enjoyment of learning mathematics.

Advantages of using GeoGebra

GeoGebra operates on the Java framework, making it compatible with Windows, Mac, Linux, or Solaris systems.

It is available free online and also as an offline version especially valuable for developing countries. Furthermore, it supports multiple languages.

The application enables users to plot points, draw vectors, segments, lines, polygons, and conic sections, and measure the attributes of geometrical objects. Dynamic features allow for exploration of properties, making and testing of conjectures relating to attributes of geometric objects and relations between them. We can also make rigid transformation and dilation of figures; draw 3D geometrical objects, nets of polyhedrons, and cross-sections of 3D objects. We can graph a function given its equations by using commands and observe how changing parameters of a function affects its graph.

Other key features include an authoring tool for creating interactive learning resources as web pages and an active international community providing technical support and teaching materials. With over a million existing GeoGebra Activities and Books, educators can initiate interactive and lively classroom discussions.

How to Utilize GeoGebra

GeoGebra provides various resources and functionalities for teachers. Educators can copy and edit existing GeoGebra resources to create personalized versions, as demonstrated in this instructional video: Link to Video.

Teachers can also find or create formative assessments for practice, using webinars for guidance: Link to Webinars.

For classrooms equipped with Chromebooks or laptops, GeoGebra allows teachers to assign interactive tasks, monitor live progress, ask questions, and facilitate dynamic discussions. A comprehensive webinar on this can be found here: Link to Webinar.

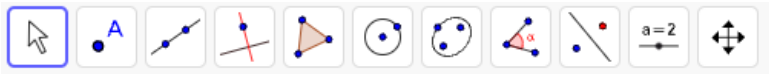
In areas with limited resources, if teachers have internet access, they can use GeoGebra for teaching and ask students to practice activities at home. Due to COVID-19, most students in such areas have access to some device at home.

Introduction to GeoGebra Interface

Let's get started with GeoGebra!

We'll be using the Classical Version 6.0.779. You can download it from the GeoGebra website: www.geogebra.org.

When you open it, it will show the toolbar given below, an Algebra view and a Graphic view. If you click on a tool, its dropdown menu will show many tools and you can select the one you need.



For instance, if we click on the third icon, the dropdown menu will show many tools e.g. **Line tool, Segment tool, Ray tool.** When we select one of these tools, the instructions for using it will appear below for a few seconds.
All the objects that we draw using these tools are automatically labeled. We can change their name by right clicking the object and selecting **Rename** from the pop up window.

If we activate in the **Graphic Area** by clicking it. It will show the toolbar given below

We can insert/delete the axes and grid by toggling and from it.

We can also change the distance between the markings on the x-axis, and y-axis and choose between major or minor grid lines by selecting .

For an introduction to the GeoGebra interface, watch the video: https://youtu.be/RYGBhRN9oHQ and https://youtu.be/IrSvVxDvSV0?t=7.

We can correct mistakes step-by-step using the Undo and Redo buttons located in the upper right corner of the window.

For changing object properties such as name, color and its opacity, the thickness of lines and size of points and style of lines and points, watch the video: https://youtu.be/DrCkzrd_MTw

For saving constructions online that we may want to edit later, watch the video: https://youtu.be/qtUFm19aAXI?list=PLqZ0eZtMcAlugmcomSSvjPBfewVbX35L7&t=7

For exporting a construction to a Google document or word, click ☰ at the top and select **Export Image**, then select **Copy to Clipboard** in the pop up window. The construction will be copied and you can paste it anywhere.

The Algebra View

All objects that we construct will be listed with their names, and some measures of these objects e.g. coordinates of points, lengths of line segments, areas of polygons, equation of lines or curves.

We can also type commands in Algebra View to plot points, draw figures and graphs of functions,and find special points relating to figures and graphs of functions instead of using the toolbar.
Watch the video: https://youtu.be/N2p4ej8pgTg?t=15 for how to use commands.

In this monograph, we describe how to use GeoGebra for

1. Drawing and measuring geometrical objects.
2. Demonstration of properties of lines, angles, and polygons.
3. Drawing figures with specified measures.
4. Rigid transformation of figures which preserve size and shape viz. reflection, translation and rotation of figures.
5. Proving congruence of geometrical objects by using rigid transformations.
6. Construction of figures using **Move, Point, Line, Segment, Ray** and **Compass** tools akin to constructions with pencil, straightedge and compass on paper in Euclidean Geometry.
7. Dilation of figures that preserve the shape but may not preserve the size

8. Proving the similarity of geometrical objects by using dilation or dilation and one or more rigid transformations.
9. Finding values of trigonometric ratios and inverse trigonometric functions and drawing their graphs.
10. Demonstration of the properties of angles, chords, secants and tangents in a circle.
11. Coordinate Geometry:
    (i) Plot a point given its coordinates.
    (ii) Find the coordinates of the midpoint of the line segment.
    (iii) Find the coordinates of a point that divides a line segment in a given ratio.
    (iv) Draw a line segment given the coordinates of two points on it.
    (v) Draw lines given its equation in slope-intercept, standard or point and slope form.
    (vi) Draw a graph of any function given its equation.
12. Finding/verifying areas and perimeters of 2D figures.
13. Drawing 3D figures and nets of polyhedrons, finding the cross-sections of 3D figures by a plane.
14. Finding/verifying surface areas and volumes of polyhedrons, cylinders, cones and spheres.

Exercises are provided for exploring and practicing these.

Have fun exploring GeoGebra!

2. Drawing Figures and Measuring Their Attributes

Plotting Points

We can plot a point by activating and selecting **Point** from its dropdown menu and clicking in the **Graphic View**. A point will be plotted there. If we want to plot a point with certain coordinates, we can use the Grid and axes or type its coordinates within parentheses and press **enter**.

We can plot a point on any object e.g. a line segment, circle, by activating and selecting **Point on Object** from its dropdown menu and clicking at a point on the object in the **Graphic View**. A point will be plotted there.

Drawing Lines

We can draw a line, ray, line segment, a line segment with a given length, vector or a vector starting from a point by activating the **Line** tool and selecting it from its dropdown menu and following the directions that show up.

We can draw a line perpendicular to a given line from a point on/outside it, the perpendicular bisector of a line segment, or a line parallel to a given line from a point outside it by activating the **Special Line** tool and selecting **Perpendicular Line, Perpendicular Bisector** or **Parallel Line** from its dropdown menu, and following the directions that show up.

Draw Angles

We can draw an angle by activating tool and plotting or selecting three points in a clockwise direction. An angle will be shown at the vertex,we can draw rays through the vertex(the point selected second) and other two points to show the angle.

We can draw the bisector of an angle by activating tool and selecting **Bisector of Angle** from its dropdown menu and click on the rays forming an angle. The bisector of the angle will be drawn.

Draw a regular polygon

Activate tool and select **Regular Polygon** from its dropdown menu, select/ plot two points and enter the number of vertices in the pop up menu. It will draw a regular polygon with the number of specified vertices.

Draw a polygon

Activate tool and select **Polygon** from its dropdown menu and click where you want a vertex, drag the mouse to the next vertex, and continue in the same manner for all the vertices and then the first vertex again. It will name all the vertices and draw a polygon with those vertices.

Draw a circle given its center and a point on the circle or radius

Select the **Circle with Center through Point** or **Circle, Center and Radius** from the dropdown menu of tool and select or plot the center and a point on the circle for circle through a point or specify the radius in the popup menu for circle with a given radius. A circle with the specified center and passing through the specified point or a circle with specified radius will be drawn.

Copy a circle

Activate the **Compass** tool from the dropdown menu of tool and select the center and a point on the circle. A circle will be copied that you can move anywhere in the construction area.

Measuring Attributes of Figures

We can find the coordinates of a point, measures of a line segment, the horizontal and vertical shifts for a vector, measure of an angle, area of a polygon, or circle by selecting them, clicking on and then selecting **Name and Value** from the pop up window.

We can also activate tool and select **Distance** and click on two points, to find the distance between them, and click on a point on one side of the angle, vertex and a point on the other side of the angle in clockwise direction to find measure of an angle, select **Area** and click inside the polygon or circle to find its area. We can also find the slope of a line by activating

tool and selecting **Slope** from its dropdown menu. These will be shown in the **Graphic View**.

Many attributes e.g. coordinates of points, length of line segments, area of the figure, equation of a line or curve when we plot or draw them are listed in the **Algebra View.**

Exercise 2: Drawing Figures and Measuring Their Attributes

1. Draw
   (a) a line.
   (b) a ray.
   (c) a line segment.,
   (d) an angle.
   (e) a vector.
   (f) a vector from a point that has the same direction and magnitude as another vector.
2. Draw and find the measure of
   (a) a line segment.
   (b) an angle.
3. Find the midpoint of a line segment.
4. (a) Draw and bisect an angle of 60°.
   (b) Verify it by measuring the angles formed by the bisector.
5. Draw a line parallel to a given line from a point P outside it.
6. (a) Draw a line perpendicular to a given line from a point P on it.
   (b) Verify it is perpendicular to the line by measuring the angles formed by the perpendicular to the line.
7. (a) Draw a perpendicular bisector of the line segment.
   (b) Verify it divides the line segment into two equal parts and it is perpendicular to the line segment by measuring the line segments and the angle formed between the line and perpendicular at the midpoint.
8. Draw
   (a) an isosceles triangle.
   (b) an equilateral triangle.
   (c) a square.
   (d) a regular hexagon.
   (e) a regular octagon.
9. Plot the number of points given below and draw a polygon with those points as vertices.
   (a) 3
   (b) 4
   (c) 6
10. Draw a circle passing through a point.
11. Draw a circle with a radius of 3 cm and copy it using the **Compass** tool
12. Draw the following quadrilaterals and write the steps of construction for them:
    (a) a rhombus
    (b) a kite
    (c) a rectangle
    (d) a square
    (e) a parallelogram
    If you can't figure out how, watch the video: https://youtu.be/tNCum6ujliQ?t=336

for drawing them using properties of diagonals for different types of quadrilaterals

13. Draw an isosceles trapezoid and write the steps for drawing it.

3. Properties of Lines, Angles, and Polygons

We can demonstrate properties of lines, angles and geometrical shapes by activating the **Measure** tool and measuring them or by activating the **Move** tool and dragging a point on the figure.

Move tool

tool can be used to select points, lines, angles and geometrical shapes and change their name, color, opacity, and style. Activating the **Move** tool and clicking on an object selects it, dragging a point on the geometrical object by holding it down on that point on the object and moving it changes its value and objects dependent on it and thus the shape of the object.

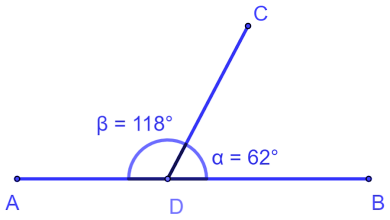
Dependent and Free Objects

An object is **free** if you can move it anywhere by activating the **Move** tool and dragging it. These points are marked in black color in the construction area.
If we mark a point on a line, circle or curve and drag it by activating the **Move** tool**,** it will only move on the line, circle or curve. These points are called **dependent** and are shown in gray color in the construction area.

The “Drag Test”

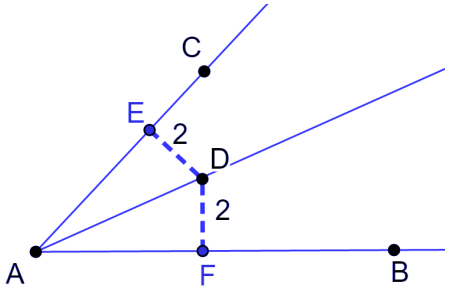
Dragging is an important way to find the geometrical properties of drawings and objects dependent on them.
If we select an object in the construction area with the **Move** tool. We can drag the object by holding it down on the object and moving it and can observe how other parts move with the selected object. For instance, a segment depends on its endpoints; when the points move, the segment must also move. For example, if we draw a line and a line perpendicular to it. Now if you select a point on the line and drag it, the line will move, so also the perpendicular but it will always remain perpendicular to it.
We can also examine if a property holds for all such figures by dragging. For example, after we have demonstrated that the sum of angles of a triangle is 180° by measuring its angles. As **Algebra View** lists all the objects with their labels, lengths, areas or other values of the attributes of objects. So if we type SumofAngles(do not leave Spaces) = $\alpha + \beta + \gamma$, where $\alpha, \beta, and\ \gamma$ are the names of the angles of the triangle in the **Algebra View**. It will give its value as 180°. If we drag a vertex, the measures of all the angles of the triangle will change, their sum will always be 180°. This demonstrates that it holds for all the triangles.

Example 1: Demonstrate that if the exterior sides of a pair of adjacent angles form a straight line(a linear pair), the angles are supplementary.



1. Draw $\overline{AB}$ using the **Segment** from the dropdown menu of tool.
2. Mark a point D on the line segment and draw a ray DC(not coincident with the line) using the **Ray** from the dropdown menu of tool starting from a point D on the line.
3. Find the measures of □CDB and□CDA, by selecting the points forming an angle in a clockwise direction. It will name the angles □α and □β and give their measures. If it does not give both the name and value, then click in the construction area, a toolbar will show, click tool and select **Name and Value** from its pop up menu.
4. In the **Algebra View** type SumofAngles = α + β. The **Algebra View** will show SumofAngles = 180°.
5. If we activate the **Move** tool and drag point C. We will notice in the Algebra view that while measures of □α and □β keep changing but the Sumofangles always remains 180°.
6. It demonstrates that for any two angles that form a linear pair, the angles are supplementary.

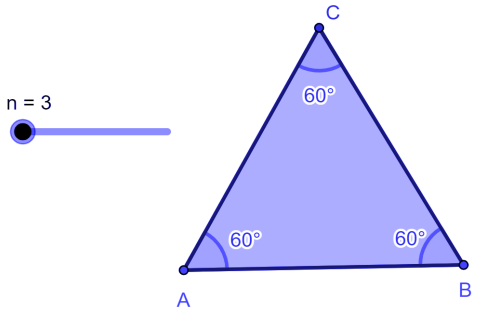
Example 2: Show that any point on the bisector of an angle is equidistant from sides of the angle.



1. Draw ∠CAB by selecting **Angle** from the dropdown menu of tool.
2. Bisect the angle by selecting the **Angle Bisector** from the dropdown menu of tool and selecting the rays forming the angle.

3. Mark a point D on the bisector using the **Point on Object** from the dropdown menu of tool.
4. Draw perpendiculars from point D to the sides of the angle selecting the **Perpendicular Line** from the dropdown menu of tool.
5. Mark the points of intersection of the perpendicular with the sides of the angle as E and F using the **Intersection** tool from the dropdown menu of tool.
6. Delete the bisectors by using the **Show/Hide** from tool or deselecting the bisectors from the Algebra View.
7. Draw $\overline{DE}$ and $\overline{DF}$ using the **Segment** tool from the dropdown menu of tool.
8. If we measure the lengths of $\overline{DE}$ and $\overline{DF}$ using the **Distance** from tool. We will notice that DE = EF.
9. If we move a point on one of the sides using the **Move** tool, we will notice that while the ∠CAB changes, DE and DF remain perpendicular to the sides and are equal. It demonstrates that it holds for any angle.
10. If we drag point D on the bisector, we will notice that DE and DF remain equal for any point D on the bisector. It demonstrates that the relation DE = DF holds for any point on the bisector of an angle.

Show that all interior angles of a regular polygon are equal.



1. Create a slider **n** for the number of vertices.

2. Activate tool and select **Regular Polygon** from its dropdown menu and then select two points and enter n for the number of vertices in the pop up menu. A regular polygon with the number of vertices shown on the slider will be drawn.
3. Activate the tool and click inside the polygon. It will show all the angles in the polygon. We notice that all of them are equal.
4. Move the point on the slider to different values of n, regular polygons with that number of sides will be drawn. All angles in those polygons for different values of n will be equal although different for polygons with different numbers of sides or vertices.
5. This demonstrates that all interior angles of a regular polygon are equal.

Exercise 3: Properties of Lines, Angles, and Polygons

1. Draw two intersecting lines. These will form 4 angles. Measure both pairs of vertically opposite angles.
   (a) What relation do you notice between the angles?
   (b) Move one of the lines. What do you notice?
   (c) What does it demonstrate?
2. Draw two parallel lines and a transversal intersecting them.
   (a) Name all the pairs of corresponding angles.
   (b) Name all the pairs of alternate interior angles
   (c) Name all the pairs of alternate exterior angles.
   (d) Interior angles on the same side of the transversal.
3. In the figure drawn in Q.2, demonstrate the relations in the measures of the following pairs of angles:
   (a) corresponding angles
   (b) alternate interior angles
   (c) alternate exterior angles.
   (d) Interior angles on the same side of the transversal
   (e) Move the first line drawn, by using the **Move** tool. What do you notice?
   (f) Move the transversal using the **Move** tool. What do you notice?
4. Demonstrate: Lines which are parallel to a given line are parallel.(Hint: Draw two lines parallel to a line and show that slopes of the two lines drawn are equal).
5. Demonstrate: If a point is on the perpendicular bisector of a line segment, then it is equidistant from the endpoints of the segment.
6. Demonstrate: The measure of each exterior angle of a triangle is equal to the sum of its two remote interior angles.
7. Interact with the first applet given in the link given below and answer the questions that follow. https://www.geogebra.org/m/munhXmzx#material/mMkZtTtJ.
8. (a) Draw a △ABC.
   (b) Find the midpoints E and F of sides AC and BC respectively by selecting the **Midpoint or Center** tool from the dropdown menu of tool and join points E and F by using the **Segment** tool.
   (c) Measure the length of $\overline{EF}$ and $\overline{AB}$. What relation do you notice between them?
   (d) Move the vertices of the ΔABC, what do you notice?
   (e) Is EF || AB? If yes, how do you know?
   (f) What does it demonstrate?
9. Draw the triangles given below and find the circumcenter (point of concurrence of perpendicular bisectors of all sides) for each triangle. Move a vertex of the triangle, what do you notice? Does the circumcenter ever fall outside the triangle? Can you make conjectures about all the points of concurrence for this type of triangle when a vertex is

moved? If yes, state it.
(a) an equilateral triangle
(b) an isosceles triangle
(c) an obtuse triangle
(d) a right triangle

10. Repeat Q.11 for
    (a) a centroid (point of concurrence of medians of all sides).
    (b) an orthocenter (point of concurrence of altitudes from all vertices to opposite sides).
    (c) an incenter (point of concurrence of bisectors of all angles).
11. Demonstrate: Perpendicular bisectors of sides of a triangle intersect in a point that is equidistant from the vertices of the triangle.
12. Demonstrate: Bisectors of angles in a triangle intersect in a point that is equidistant from the sides of the triangle.
13. Draw a triangle and circumscribe a triangle that passes through its vertices.
14. Draw a triangle and inscribe a circle in it that touches its sides.
15. Name all the special quadrilaterals that are parallelograms.
16. Draw a parallelogram ABCD, move point A so that □DAB is 90°. What type of special quadrilateral results?
17. Draw a parallelogram ABCD, and move point D so that AD = AB. What type of special quadrilateral results?
18. Draw an isosceles trapezoid ABCD so that the lengths of its non-parallel sides AD and BC are equal.
    (a) Give steps of construction.
    (b) Measure its base angles. What relation do you notice between their measures?
    (c) Measure its diagonals. What relation do you notice between their measures?
    (d) Label the midpoints of $\overline{AD}$ and $\overline{BC}$ as E and F respectively. What relation do you notice between measures EF and AD + BC?
    (e) Is $\overline{EF}$ parallel to $\overline{AB}$? If yes, explain how you know.
19. Draw a parallelogram and demonstrate that
    (a) its opposite sides are equal.
    (b) its opposite angles are equal.
    (c) its diagonals bisect each other.
    (d) its consecutive angles are supplementary.
    (e) move a vertex by activating the **Move** tool. What do you notice?
    (If you can't figure out how, watch the video: (https://youtu.be/yzt7q-MAVvQ?t=96)
20. Draw a rectangle and demonstrate
    (a) its diagonals are equal and bisect each other.
    (b) move a vertex by activating the **Move** tool. What do you notice?
21. Draw a square and demonstrate
    (a) its diagonals are perpendicular bisectors of each other.
    (b) move a vertex by activating the **Move** tool. What do you notice?
22. Draw a rhombus and demonstrate that

(a) its sides are congruent.
(b) its diagonals bisect the angles.
(c) its diagonals are perpendicular bisectors of each other.
(d) move a vertex by activating the **Move** tool. What do you notice?

23. Draw a kite and demonstrate
    (a) its diagonals are perpendicular bisectors of each other.
    (b) One of the diagonals bisects a pair of opposite angles.
    (c) One pair of opposite angles is congruent.
    (d) Move a vertex by activating the **Move** tool. What do you notice?
24. Watch the video and describe what it demonstrates about the sums of interior and exterior angles of polygons with n sides.
    https://www.youtube.com/watch?v=qyPnFjzHek0

4. Drawing Figures with Specified Measures

We can draw figures with specified measures by using **Segment with Given Length, Angle with Given Size, Circle with Center & Radius** tools for line segments, angles, circles and a combination of these for triangles and quadrilaterals.

Hide lines, rays, or circles we don't want to show

Sometimes we need to draw lines, rays or circles that we don't want to show after drawing the figure, we can hide them by right clicking them and selecting **Show/Hide Object** from its dropdown menu or deselecting the line, ray or circle from the **Algebra View.**

Slider

For drawing many figures with specified measures, we can use sliders, as by moving the point on the sliders, we can draw many geometrical objects with any values of that attribute within the range specified on a slider. We can use many attributes of the geometrical object by using their sliders. For example if we want to draw many triangles whose three sides are given, we can have sliders for the length of each side by moving the point on the sliders of those sides, we can have the specified length.

A slider is the graphical representation of a free number or free angle. For how to create and use a slider watch the video: https://youtu.be/KO1xaSFdHSA?t=351

Draw a line segment with a given length

1. Create a slider **a** for the length of a line segment.
2. Activate tool and select **Segment with a Given Length** from its dropdown menu and enter **a** for length **in the pop up window.**
3. A line segment of length **a** equal to the value shown on the slider will be drawn. We can change the value of **a** by moving the point to any other value of **a** to draw a line segment with that length.

Draw an angle with a given measure

1. Create a slider α for an angle.
2. Draw a ray through point A and mark a point B on it.

3. Activate tool and select **Angle with Given Size** from the dropdown menu of the tool.
4. Select leg point B and vertex A and then enter α for the angle and counterclockwise in the pop up window. An angle with measure α shown on the slider will be drawn at the vertex A and a point B' will be shown.
5. Draw a ray beginning at the vertex A and passing through point B'.Then □B'AB is the angle with size α.
6. By moving the point on the slider, we can change the size of the angle to any desired measure.

If we want the angle to look nice and draw arrows at the endpoint to show that rays tend to infinity. We can select points on the ray, where we want to draw arrows and draw segments from the vertex to those points and hide the rays. Then select the segment and click on the third line and select → from the pop up window.

Draw a Circle with a Given Radius

1. Create a slider **a** for the length of the radius.
2. Activate tool and select **Circle with Center & Radius** from the dropdown menu of the **Circle** tool. Select/plot the point for the center and enter **a** for the radius in the popup menu. A circle with the radius shown on the slider will be drawn.
3. You can draw any circle with that radius by changing to that value on the slider.

Draw triangles with given measures

For all such triangles we will need to draw line segments of a given length, an angle with a given measure, a circle with a given radius, and plot and label the point of intersection of two lines, a line and a circle or two circles.
A triangle whose shape and size can't be changed by dragging its vertices is called **rigid**. However, we can change the location and orientation of a rigid triangle.
A rigid triangle can be drawn if we know any of the following three measurements that satisfy the properties of a triangle viz. the sum of angles of a triangle is equal to 180° and the sum of lengths of two sides of a triangle is greater than the length of the third side.

- The length of all its sides(SSS)
- Two sides and an angle included between them(SAS)
- Two angles and a side included between them(ASA)
- Two angles and one corresponding side.(AAS)

Draw a triangle given the length of all its sides(SSS)

1. Construct a triangle ABC with AB = 5 cm, AC = 3.5 cm, and BC = 4.3 cm.
2. Draw a line segment AB with a length equal to 5.
3. Draw a circle with center A and radius 3.5.
4. Draw a circle with center B and radius 4.3
5. Mark the point of intersection of the two circles drawn, it will be labeled C.
6. Draw line segments AC and BC. Triangle ABC will be drawn with specified measures of its sides. Hide the circles by right clicking the circle and selecting **Show Object** from its dropdown menu or deselecting the circle from the **Algebra View.**
7. You can check the lengths of sides by measuring them.

Draw a triangle given two sides and an angle included between them(SAS)

Draw a triangle ABC with AC = 4 cm, AB = 5 cm and $\angle$BAC = 50°.

1. Draw a line segment AB with a length equal to 5.
2. Draw □A = 50°
3. Draw a circle with center A and radius 4.
4. Mark the point of intersection of the circle with the side of the angle, it will be labeled C..
5. Draw line segments AC and BC. Hide the circle and ray.
6. Triangle ABC is the triangle with given measures.

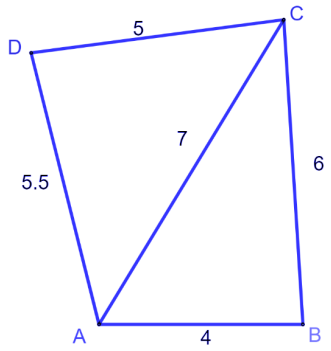
For special triangles that have certain relations between sides or angles, we can draw them when fewer measurements are known e.g. for an equilateral triangle if only a side is given, for an isosceles triangle if length of base and angle made by the equal sides with the base is given.

Draw quadrilaterals with given measurements

A rigid quadrilateral can be drawn if we know any of the five measurements given below:

- The lengths of its four sides and an angle
- The lengths of its four sides and a diagonal
- Two diagonals and three sides
- Two adjacent sides and three angles
- Three sides and two included angles are given.

Example 1: Construct a quadrilateral ABCD where AB = 4 cm, BC = 6 cm, CD = 5 cm, AD = 5.5 cm, and AC = 7 cm.



When four sides and a diagonal are given, we first draw a triangle with two sides and a diagonal and then find the fourth point.

1. Activate tool and select **Segment with Given Length** from the dropdown menu of the tool, and select a point and enter 4 for the length of the line segment. A line AB with a length of 4 cm will be drawn
2. Activate tool and select **Circle with Given Radius** from the dropdown menu of the tool. Select A for the center and enter 7 for the radius. A circle with center A and radius 7 will be drawn.
3. Activate tool and select **Circle with Given Radius** from the dropdown menu of the tool. Select B for the center and enter 6 for the radius. A circle with center B and radius 6 will be drawn intersecting the circle with center A.
4. Activate tool and click on the point of intersection of the circles. It will be labeled as C.
5. Activate tool and select **Line Segment** from its dropdown menu. Join points A and C and B and C and hide the circles. It will show triangle ABC.
6. Activate tool and select **Circle: Center & Radius** from the dropdown menu of the tool. Select A for the center and enter 5.5 for the radius. A circle with center A and a radius of 5.5 cm will be drawn.
7. Activate tool and select **Circle with Given Radius** from the dropdown menu of the tool. Select C for the center and enter 5 for the radius. A circle with center C and radius 5 will be drawn intersecting the circle with center A.

8. Activate tool and select **Intersection** from the dropdown menu of the tool. Click on the point of intersection of the circles. It will be labeled as D.
9. Activate tool and select **Segment** from its dropdown menu. Join points A and D and C and D and hide the circles.
10. The quadrilateral ABCD is the quadrilateral with specified measures.

Draw Special Quadrilaterals with Specified Measures

For special quadrilaterals that have certain relations between sides or angles or diagonals, we can draw them when fewer measurements are known.
If we want to draw many of these with different measures, we can draw one by creating sliders for those measures and by moving the point on the slider, we can draw many of these with measures within the range of the sliders.

Draw a rectangle whose length and width are given

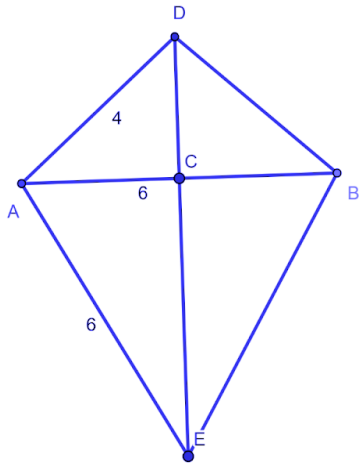
1. Create a slider **a** for the length of the rectangle.
2. Draw a line segment AB with length **a**.
3. Draw perpendicular lines to the line segment at A and B.
4. Create a slider for **b** .
5. With A and B as centers, draw circles with radii equal to the width **b** .
6. Mark the points where the circles intersect the perpendicular lines and label these points as C and D.
7. Activate **Line Segment** tool and join the points A and C, C and D, and D and B Activate the **Show/Hide Object** tool and hide the lines.
8. ABCD is the required rectangle.
9. You can move the point on the slider to make a rectangle of any length and width within the range specified for the sliders.

Draw a parallelogram the measure of whose sides are a and b, and the angle between the sides is θ.

1. Create a slider for **a**.
2. Draw a $\overline{AB}$ with length **a**.
3. Create a slider for angle θ.
4. Draw an angle at A with a measure of θ.
5. Create a slider for **b**.
6. With A as a center and radius **b** draw a circle.
7. Mark the point where the circle intersects the ray of the angle and label it as D.

8. Draw a line parallel to AB and passing through D.
9. With B as the center and radius **b** draw a circle.
10. Mark the point of intersection of the circle with the parallel line drawn and label it as C.
11. Activate **Line Segment** tool and join the points B and C, C and D. Hide the circles and lines that you don't want to show..
12. ABCD is the required parallelogram.
13. You can move the points on the slider to make a parallelogram of desired length of adjacent sides and angle between them..

Draw a kite one of whose diagonals is 6 cm, the shorter side is 4 cm and the longer side is 6 cm.



1. Draw a line segment AB with a length of 6 cm.
2. Draw a perpendicular bisector of the $\overline{AB}$.
3. With A as the center and radius 4 draw a circle. Label its point of intersection with the perpendicular bisector as D and hide the circle.
4. With A as the center and radius 6 draw a circle. Label its point of intersection with the perpendicular bisector on the opposite side as E and hide the circle.
5. Join AE, EB, BD, and DA.
6. AEBD is the kite with the given measures.

Exercise 4: Drawing Figures with Specified Measures

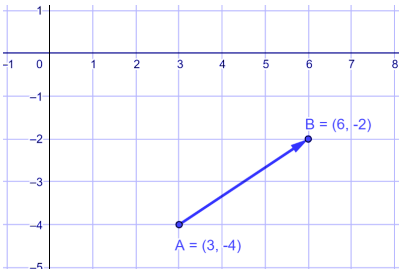
1. Draw a line segment of length 4 cm.
2. Draw an angle with a measure of 75°.
3. Draw a regular hexagon whose side is 3 cm.
4. Draw a circle whose radius is 2 cm.
5. Draw triangles whose measures are given below. If a triangle cannot be constructed, state the reason for it. For each of the triangles you could draw, try to change the size, orientation or location of the triangle. What do you notice?
   (a) AB = 5 cm, AC = 4.5 cm, and BC = 6.3 cm.
   (b) AB = 4 cm, BC =3 cm , and CA = 8 cm.
   (c) AB = 4 cm, AC = 4.5 cm, and BC = 4.5 cm.
   (d) AB = 6 cm, AC = 5 cm and $\angle$BAC = 60°
   (e) AB = 3 cm, AC = 5 cm and $\angle$BAC = 90°
   (f) AB = 4 cm, AC = 5 cm and $\angle$BAC = 120°
   (g) AB = 6 cm, $\angle$A = 50° and $\angle$B = 75°
   (h) AB = 6 cm, $\angle$A = 50° and $\angle$B = 80°
   (i) AB = 5 cm, $\angle$A = 90° and $\angle$B = 90°
   (j) AB = 6 cm, $\angle$A = 60° and $\angle$B = 60°
   (k) AB = 6 cm, $\angle$A = 120° and $\angle$B = 90°
6. Draw quadrilaterals whose measures are given below and for each of the quadrilaterals try to change the size, orientation or location of the triangle. What do you notice?
   (a) Draw a quadrilateral ABCD where AB = 4, BC = 3, CD = 2, DA = 3.5 and $\angle$A = 60°.
   (b) Draw a quadrilateral ABCD that has AB = 3.5 cm, BC = 6.5 cm, $\angle$A = 75°, $\angle$B = 105°
   and $\angle$C = 120°.
   (c) Draw a quadrilateral ABCD that has AB = 3.5 cm, BC = 6.5 cm, $\angle$A = 75°, $\angle$B = 105°
   and $\angle$C = 120°.
   (d) Draw a quadrilateral ABCD, given that BC = 4.5 cm, AD = 5.5 cm, CD = 5 cm the diagonal AC = 5.5 cm and diagonal BD = 7 cm.
   (e) Draw a parallelogram whose adjacent sides are 5 cm and 4 cm and the angle between the two sides is 50°.
   (f) Draw a rhombus whose diagonals are 6 cm and 4 cm.
   (g) Draw a trapezoid ABCD in which AB || CD, AB = 4 cm, CD = 3 cm, and, m$\angle$A = 70°.

5. Transformations of Figures

In GeoGebra, the **Transformation** tools are grouped under tool. We assume by now you have become familiar with most of the tools and how to use them. The icons for the tool unless it is new will not be given in instructions.

Vectors

A vector is a quantity that has both length and direction. You can think of a vector as a directed line segment. The vector below may be named AB or v

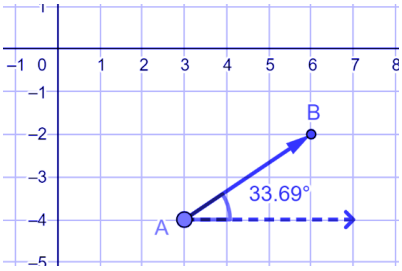


A vector can also be named using component form: Vector AB = (3,2), where the first component gives the horizontal shift from A to B (6 - 3 = 3) units and the second component gives the vertical shift from A to B (-2 - (-4)) = 2 units.

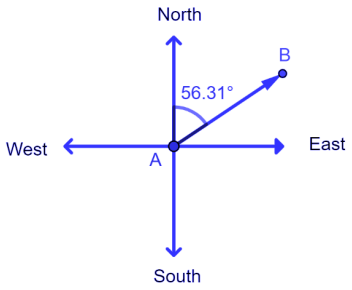
In Algebra View, a vector is given in vertical form within parentheses with horizontal shift at the top and vertical shift below that.

The **magnitude of a vector** is given by its length.

The **direction of the vector** is the angle that it makes with the horizontal line. This angle is measured in a counterclockwise direction from the positive x-axis. The direction of vector AB is 33.69°.



The direction of a vector can also be given as a bearing relative to the compass directions **north, south, east and west.** Vector AB has a bearing of N 56.31°E.



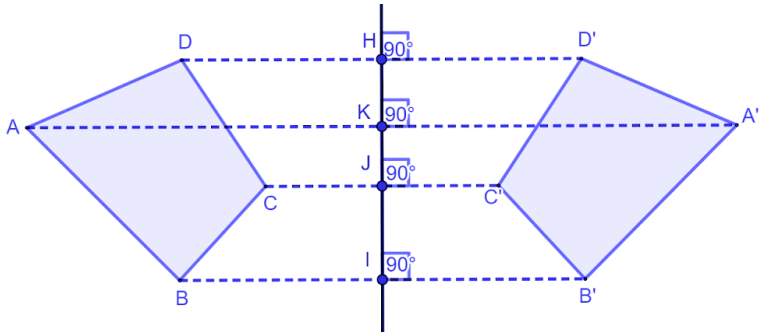
Transformation of Figures

Transformations are changes done in the shapes on a coordinate plane by rotation, reflection or translation which preserve both the shape and size of the figure.
Watch the video: https://youtu.be/JHvXL5Fph0M?t=19 to learn how to translate, reflect, and rotate figures by using GeoGebra.
You can explore these transformations of figures yourself or make them by following the instructions given below:

Reflect a polygon about a line

1. Draw the figure to be reflected and a line about which the figure is to be reflected.
2. Select **Reflection** from the dropdown menu of the **Transformation** tool. Select the figure and the line. The figure will be reflected.

Example 1: A quadrilateral and its reflection about a line is given below. H, K, J, I are The midpoints of $\overline{DD'}$, $\overline{AA'}$, $\overline{CC'}$, and $\overline{BB'}$.



(a) What do you notice about the line segments joining the corresponding vertices and the line reflection?
(b) Find the mid-points of the line segments joining the corresponding vertices. What do you notice?
(c) Can you use the properties noticed in (a) and (b) to draw a reflection of a figure with paper and pencil? If yes, explain how.
(d) If you change the figure by activating the **Move** tool and dragging a point on the figure what do you notice?,

(e) If you change the line by activating the **Move** tool, what do you notice?
(f) What else do you notice?

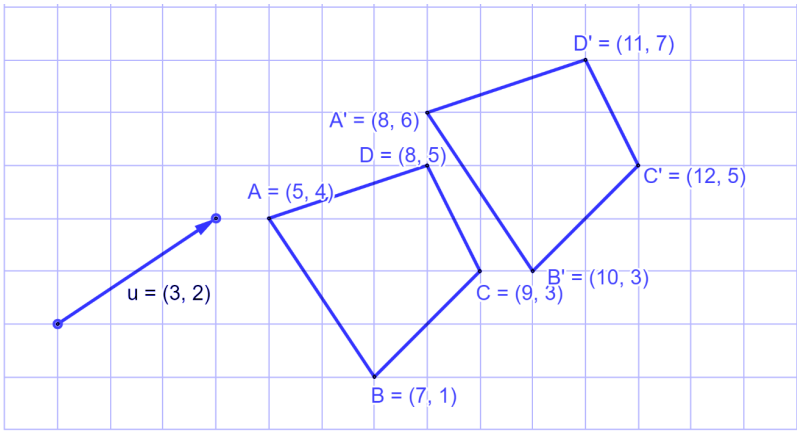
Answer

(a) We notice that the line of reflection is perpendicular to all the line segments joining the corresponding vertices of the preimage and image.
(b) The midpoints of all the line segments joining the corresponding vertices of pre-image and Image lie on the line of reflection.
(c) Yes. We can find the corresponding vertices of the image from the vertices of the pre-image by drawing a line perpendicular to the line of reflection from the vertices of the pre-image and selecting points on them at a distance equal to the distance from the corresponding vertices of the pre-image to the line of reflection. Joining these vertices by line segments as in the pre-image will give the image.
(d) If we activate the **Move** tool and drag a vertex to change the figure. Then we will notice that the reflected figure changes accordingly.
(e) If we activate the **Move** tool and drag a point on the line of reflection to change the line. Then we notice that the figure would be reflected about this line.
(f) We also notice that the shape and size remain the same after reflection about a line, however, the orientation (lettering of the vertices of the image) changes from clockwise to counterclockwise.

Translate a polygon by a vector

1. Draw a polygon. To ensure the coordinates of its vertices will be given, activate
   and select **Name and Value** from the dropdown menu of tool.
   If we want to restrict figures to the integral values only in a transformation, we can do it by activating in the toolbar in the graphic area, selecting **Grid** from it and then clicking **Fixed to Grid** in the dropdown menu of **Point Capturing**.
2. Draw a vector. It will be shown as an arrow in the **Graphic view** and as a vector u with two components, the one on the top denoting the horizontal shift and one below that the vertical shift in the **Algebra View**.
3. Activate the **Transformation** tool and select **Translate by a Vector** from its dropdown menu. Then select the figure and the vector by clicking inside the figure and on the vector. The figure will be translated by the vector.

Example 2: The figure given below gives a quadrilateral and its image when it was translated by vector (3,2), where 3 and 2 denote the horizontal and vertical shifts.



(a) What do you notice about the coordinates of the vertices of the image and preimage? Can you use them to draw the image? If yes, explain how.
(b) Change the vector by dragging a point on it. What do you notice?
(c) Change the figure by dragging its vertices. What do you notice?
(d) What else do you notice?

Answer

(a) The coordinates of the corresponding vertices of the quadrilateral and its image show that the x and y-coordinates of each corresponding vertex are x + 3, and y + 2 respectively, where (x,y) is the coordinate of a vertex of the preimage and 3 and 2 denote the horizontal and vertical shifts of the vector. It demonstrates that we can plot the corresponding vertices of the image of the figure by adding the values of horizontal and vertical shifts given in the vector to the values of the x and y-coordinates of the original figure and plotting and joining them by line segments. We can use this relation for translating a figure by using graph paper and pencil.
(b) If we change the vector by dragging a point on it to create a new vector. The figure is translated by new values of the horizontal and vertical shifts given by the vector.
(c) If we change the figure by dragging one or more of its vertices to create a new polygon. The new polygon is translated by the vector.
(d) We also notice that the shape, size, and orientation remain the same after translation by a vector.

Rotate a polygon by an angle around a point of rotation

1. Draw a polygon and plot a point.
   Type 0deg in the Algebra view, a slider for α the angle of rotation will be created or activate the tool and select **Slider** from its dropdown menu and name the angle as α.

2. Activate **Rotate about a Point** from the dropdown menu of the **Transformation** tool. Select the polygon by clicking inside it and the point and enter α for the angle in the pop up menu. A rotated image about the point by angle α will be shown. You can move the slider to change α the angle of rotation.

Example 3: A quadrilateral ABCD and its rotation about a point E in the exterior of the quadrilateral ABCD by an angle of 70° is given below. The color and style of the sides of these angles were changed by selecting the **Move** tool and selecting a different color and line style in the **Toolbar** in the settings.

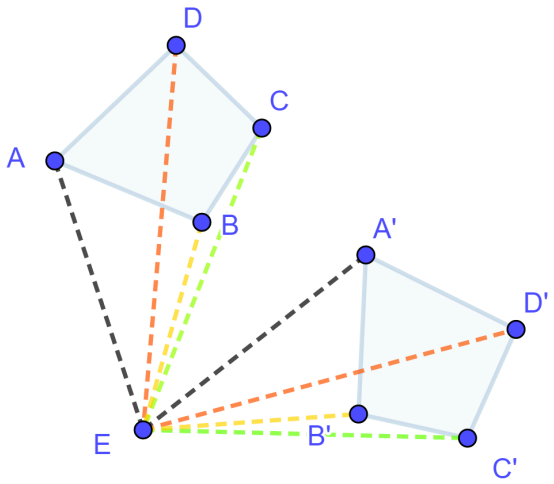
(a) Find □AEA', □BEB', □CEC', and □DED'.(We can find these by activating the **Measure** tool and clicking inside the figure or from the **Algebra View)**. What do you notice?

(b) Find the lengths of line segments between E and corresponding vertices i.e. AE and E'A; BE and BE'; CE and CE'; and DE and DE'. (We can find these by activating the **Distance** tool and clicking the segments or from the **Algebra View)**. What do you notice?

(c) Drag a vertex of the quadrilateral, what do you notice?

(d) Can you use properties (a) and (b) to draw a rotated image using paper, pencil and protractor of the pre-image by a certain angle? If yes, explain how.

(e) What else do you notice?



Answer

(a) The □AEA' = □BEB' = □CEC' = □DED' = 70° (the angle of rotation). We can also check it in the Algebra View.

(b) AE = AE'; BE = BE'; CE = CE'; and DE = DE'.

(c) If you drag a vertex of the quadrilateral its shape changes, and so does its image but the measures of angles □AEA', □BEB', □CEC', and □DED' and the lengths of the line segments between corresponding vertices and E remain equal for all vertices.

(d) The properties given in (a) and (b) can be used to draw a rotated image of a polygon on paper by plotting the corresponding vertices as follows:
For finding point A' the point corresponding to point A of the pre-image, join the vertex A to point E about which the figure is to be rotated, make an angle at E using AE as one side equal to the angle by which we want to rotate and then mark a point A' on the other side of the angle at a distance equal to the length of the segment AE. Similarly find the corresponding vertices B', C', and D' of vertices B, C, and D respectively. Join A'B', B'C', C'D', and D'A' to form quadrilateral A'B'C'D' which is the image of quadrilateral ABCD.

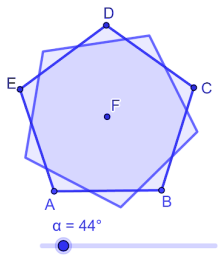
(e) We also notice that the shape, size, and orientation remain the same after rotation about a point.

Find the angles for which a regular polygon maps onto itself when rotated about the center of the polygon by those angles.

1. Draw a regular polygon and mark its center by activating the **Midpoint or Center** tool.
2. Create a slider for angle α.
3. Activate **Rotate around Point** tool from the dropdown menu of the **Transformation** tool. Select the polygon by clicking inside it and then its midpoint. Specify the angle of rotation as α. The figure will rotate about the point by an angle α.
4. Change the values of α on the slider. Note the values of α for which the polygon maps onto itself.

Example 4: What is the smallest angle of rotation that will map a regular pentagon onto itself when rotated about the center of the pentagon.

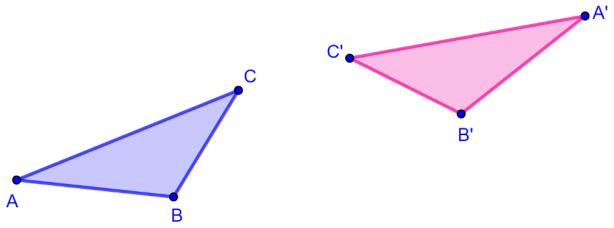
1. Create a slider for angle α .
2. Draw a regular pentagon ABCDE.
3. Find the center of the polygon.It will be automatically named as F.
4. Rotate the pentagon by angle α about point F. The pentagon and its image after rotation by α is given below:



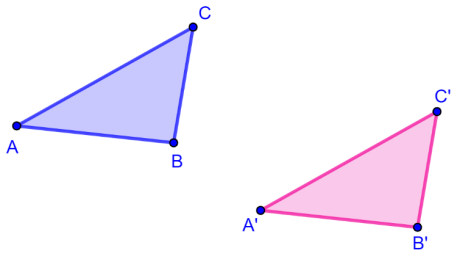
5. As we move the point on the slider for α , we notice that the pentagon maps onto itself, when α is 72°, 144°, 216°, 288°, and 360°. Thus the smallest angle when the pentagon maps onto itself is 72°.

Exercise 5: Transformations of Figures

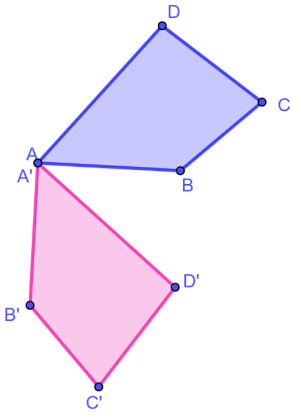
1. For each of the figures and its image given below, state whether the pink figure is a reflection, translation or rotation of the blue figure.
   (a)



   (b)



   (c)



2. Plot the following points and reflect them about the x-axis. State the coordinates of their image?
   (a) (2,3)
   (b) (-3,4)
   (c) (3,-5)
   (d) (-1,-4)

What do you notice? What will be the coordinates of the image of point (x,y) when reflected about the x-axis?

3. Plot a few points and reflect about the y-axis and note the coordinates of their image? What will be the coordinates of the image of point (x,y) when reflected about the y-axis?
4. Plot a few points and reflect about the line y = x and note the coordinates of their image? What will be the coordinates of the image of (x,y) when reflected about the line y = x?
5. Plot a few points and reflect them about the origin and note the coordinates of their image? What will be the coordinates of the image of (x,y) when reflected about the origin? **(**For reflecting about the origin, plot a point at the origin.)
6. (a) Draw a triangle and a line.
   (b) Reflect the triangle about the line.
   (c) Join the corresponding vertices, what relation do you notice between the line of reflection and these line segments?
   (d) Find the midpoints of line segments joining the corresponding vertices, what do you notice?
   (e) Change the shape of the triangle by moving a vertex, what do you notice?
   (f) Change the line by moving a point on it. What do you notice?
   (g) What else do you notice?
   (h) Draw a triangle on graph paper and a line of reflection, draw its image using the properties observed in (c) and (d).
7. (a) Draw a quadrilateral and a line.
   (b) Reflect the quadrilateral about the line.
   (c) Join the corresponding vertices, what relation do you notice between the line of reflection and these line segments?
   (d) Find the midpoints of line segments joining the corresponding vertices, what do you notice?
   (e) Change the shape of the quadrilateral by moving a point, what do you notice?
   (f) Drag the line, what do you notice?
   (e) What else do you notice?
8. (a) Draw a regular hexagon and a line.
   (b) Reflect the hexagon about the line.
   (c) Change the shape of the hexagon by moving a point, what do you notice?
   (d) Drag the line, what do you notice?
   (e) What else do you notice?
9. (a) Draw a circle and a line.
   (b) Reflect the circle about the line.
   (c) Change the size of the circle by moving a point on it, what do you notice?
   (d) Drag the line, what do you notice?
10. Draw a triangle and translate it by T(3,4). (To draw a vector with 3 and 4 as horizontal and vertical shift respectively. Plot a point A(x,y) where x and y may have any values and point B = (x+3, y+4) and draw a vector from point A to point B.
11. (a) Draw a triangle the coordinates of whose vertices are (2,2), (-2,1), and (-1,4). (You can plot a vertex of the triangle by typing its x and y coordinates with

a comma between them within parentheses in the **Algebra View**).

(b) Translate it by a vector (3,4). What is the relation between the coordinates of vertices of the original triangle and its image?

(c) Drag one of the vertices, what do you notice?

(d) Use this property to draw a triangle on graph paper and translate it by T(-3,2).

12. (a) Draw a quadrilateral and translate it by T(-2,4).

(b) Find the coordinates of the corresponding vertices of the image, what relation do you notice between the coordinates of vertices of the quadrilateral and coordinates of its image.

(c) Drag one of the vertices, what do you notice?

(d) Draw a quadrilateral on graph paper whose vertices are (-7,2), (-4,2), (-2,4), and (-5,5) and translate it by T(3,-3).

13. (a) Draw a ΔABC and plot a point P, rotate it by an angle of 80° about a point P that is in the exterior of the triangle.

(b) Join the corresponding vertices of the preimage and image and measure □APA', □BPB', and □CPC'. What do you notice?

(c) Measure the lengths of line segments PA and PA', PB and PB', and PC and PC'. What do you notice?

(d) Move the point P, do properties (b) and (c) still hold?

(e) Draw a triangle on graph paper , coordinates of whose vertices are (-4,2), (-2,0), and (0,1), and draw its image using a pencil when rotated about point P(1,-2) by 60° counterclockwise.

14. (a) Draw a quadrilateral ABCD and rotate it by an angle of 90° about a point P in the exterior of the quadrilateral.

(b) Join point P to the vertices of the preimage and its image by using the **Segment** tool and measure □APA', □BPB', □CPC' and □DPD'. What do you notice?

(c) Find the lengths of line segments PA and PA', PB and PB', PC and PC, and PD and PD'. What do you notice?

(d) Move the point P, do properties (b) and (c) still hold?

(e) Draw a quadrilateral on graph paper coordinates of whose vertices are (-7,2), (-4,2), (-3,4), and (-6,5) and draw its image by rotating it about point P(-2,-1) by 90° using a pencil and protractor

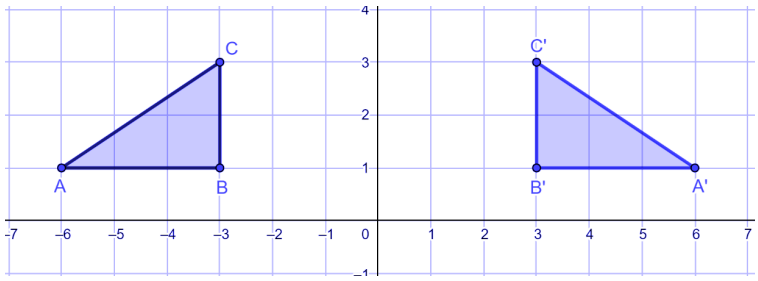
15. Find the angles of rotation that will map the following figures onto themselves when rotated about the center of the figure:

(a) an equilateral triangle.

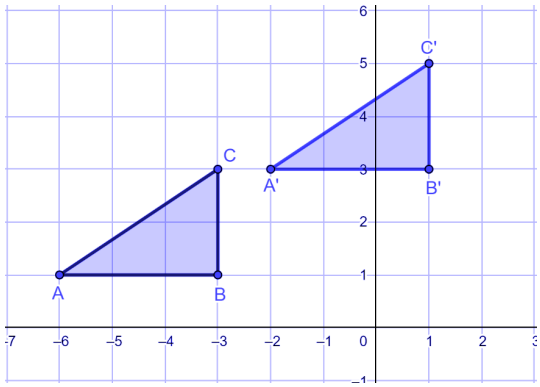
(b) a regular hexagon.

(c) a regular octagon.

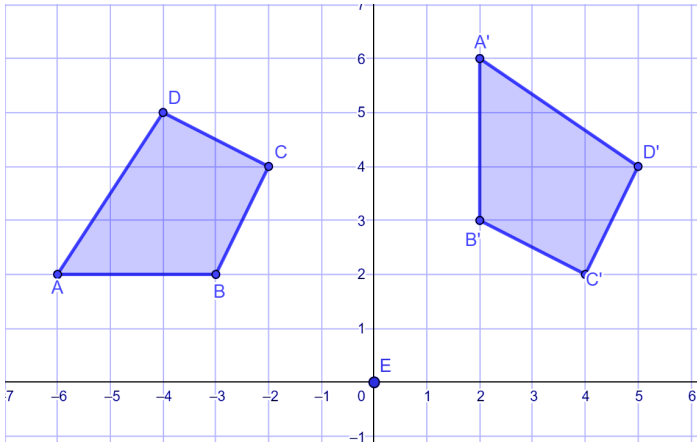
16. Triangle A'B'C' is the mirror image of triangle ABC. Write the equation of its line of reflection.



17. Triangle A'B'C' is the image of triangle ABC after a translation. Find its vector of translation or T(x,y) (horizontal and vertical).



18. Quadrilateral A'B'C'D' is the rotated image of quadrilateral ABCD about the origin. Find its angle of rotation.



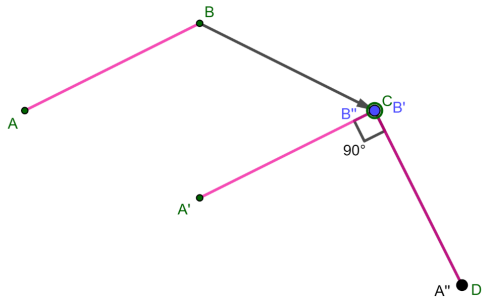
6. Proving Congruence by Rigid Transformations

We normally prove theorems by using definitions, postulates properties of algebraic equality, properties of congruence or other theorems. We can also prove them by **rigid transformations**. A **rigid transformation** is a transformation in a plane that preserves the shape and size of the figure. Reflections, translations, rotations, or a combination of these three transformations are rigid transformations. However as a dilation does not preserve size, it is not a rigid transformation.
Two figures are **congruent** if they have the same shape and size.
We can prove congruence of two figures by transforming one of the figures by trying a rigid transformation or a combination of rigid transformations to coincide with the other figure. If it coincides with the other figure, the two figures are congruent, otherwise they are not congruent.
From now on, we will assume that you know how to translate by a vector, reflect about a line, rotate about a point and will not give icons and instructions for them.

Proving Congruence of Line Segments by Transformations

Interact with Animation in https://www.geogebra.org/m/bcHwjkZ9#material/dFADRr9G and answer the questions given there.

Example 1: Are $\overline{AB}$ and $\overline{CD}$ congruent?



To prove $\overline{AB} \cong \overline{CD}$

1. Draw a vector from point B to point C.
2. Translate $\overline{AB}$ by vector BC. Its image $\overline{A'B'}$ will be drawn.
3. Measure □A'B'D which gives 90° as its measure.
4. Rotate $\overline{B'A'}$ about point C by 90° in counterclockwise direction, its image $\overline{B"A"}$ will be drawn that coincides with $\overline{CD}$. Therefore $\overline{AB} \cong \overline{CD}$.

Proving Congruence of Angles

Interact with Animation: https://www.geogebra.org/m/bcHwjkZ9#material/nnJWaaer

and prove that angle □A ≅ □B. (Hint: You will need to translate a point too, to measure the angle of rotation).

Proving congruence of triangles by transformational Geometry

For two examples for proving congruence of triangles watch the videos:
https://youtu.be/ojDwhBrKI6E?t=300
https://youtu.be/teKyr8CrUNI?t=201

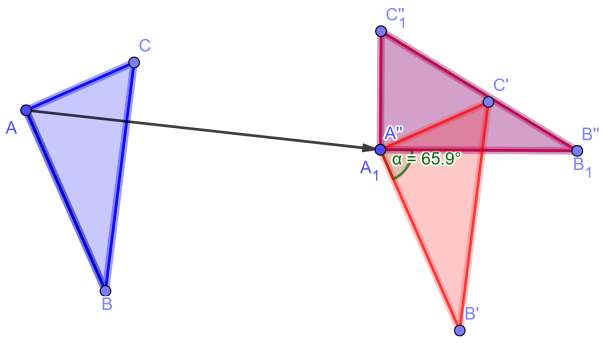
Proving Congruence of triangles with SSS or SAS or ASA equal by transformational Geometry

If two triangles are congruent their corresponding sides and angles are congruent. We don't need to have all of these to prove the congruence of two triangles.
Two triangles for which any of the following conditions are satisfied are congruent:

- If two sides and the angle included between them of one triangle are congruent to two sides and the angle included between them of another triangle, then the triangles are congruent(SAS).
- If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent(ASA).
- If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent(SSS).
- If two angles and a non-included side of one triangle are congruent to the corresponding angles and corresponding non-included side of another triangle, the two triangles are congruent(AAS).

We can prove congruence of such triangles by rigid transformations.
For a dynamic illustration of SAS congruence, watch the Video:
https://youtu.be/_pedzAC3zCY?list=PLIlF_FGjAuZVZTEFth_X9G2Gvt0nFwA9L&t=49

Example 1: In triangles $\Delta ABC \text{ and } \Delta A_1B_1C_1$ have , $AB \cong A_1B_1,\ BC \cong B_1C_1,\ \text{and } CA \cong C_1A_1$
Prove $\Delta ABC \cong \Delta A_1B_1C_1$ by transformation of ΔABC.



1. Draw a vector from A to A_1.
2. Translate triangle ABC by vector AA_1, $\Delta A'B'C'$ will be drawn.
3. Measure $\angle B_1A_1B'$, its value equals 65.9° will be shown.
4. Rotate $\Delta A'B'C'$ by 65.9°, counterclockwise. $\Delta A'B'C$ coincides with $\Delta A_1B_1C_1$.
5. Thus $\Delta ABC \cong \Delta A_1B_1C_1$.

Proving Congruence of Rigid Quadrilaterals

If two quadrilaterals are congruent their corresponding sides and angles are congruent. We don't need all of these to prove the congruence of two quadrilaterals.
Two quadrilaterals in which any of the following five measurements are equal are congruent.

- Length of its four sides and an angle.
- Lengths of its four sides and a diagonal.
- Length of its two diagonals and three sides are known.
- Length of its two adjacent sides and three angles are known.
- Length of its three sides and two included angles are given.

We can prove congruence of such quadrilaterals in a manner similar to that of triangles.

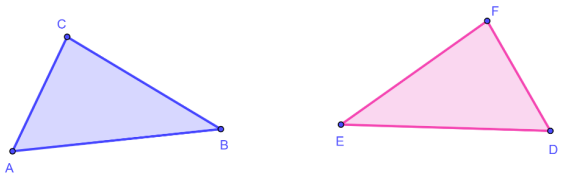
Exercise 6: Proving Congruence by Rigid Transformations

Interact with the applets given below and describe the transformations you performed to prove congruence of two triangles.

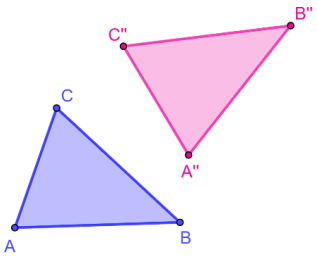
1. https://www.geogebra.org/m/bcHwjkZ9#material/wYtNhjKr
2. https://www.geogebra.org/m/bcHwjkZ9#material/ft7zHZdJ
3. https://www.geogebra.org/m/bcHwjkZ9#material/Y6UMXAZG
4. https://www.geoghttps://www.geogebra.org/m/fes47nfhebra.org/m/bcHwjkZ9#material/gByrFeJV
5. Prove that a diagonal divides a parallelogram into two congruent triangles by using rigid transformations.
6. Go to https://www.geogebra.org/m/fnb7wagz and find the figures for Q.6-Exercise 5.

   Click ⋮ and select **Open in App** and prove using rigid transformations triangles are congruent. Describe the transformations you performed to prove congruence of two triangles.

   (a)

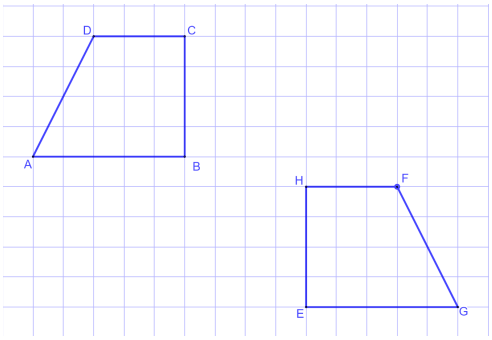


   (b)

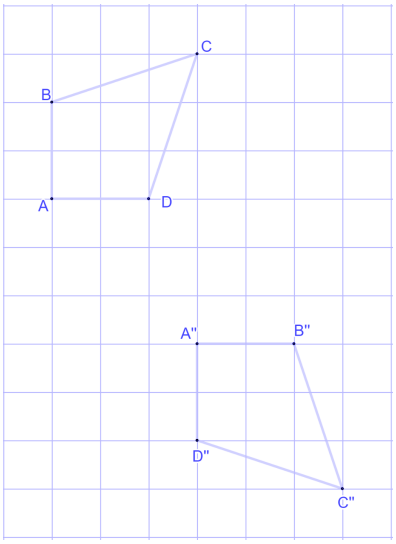


7. Are the quadrilaterals given below congruent? If yes, specify the transformation or combination of transformations that you can perform to map one quadrilateral to the

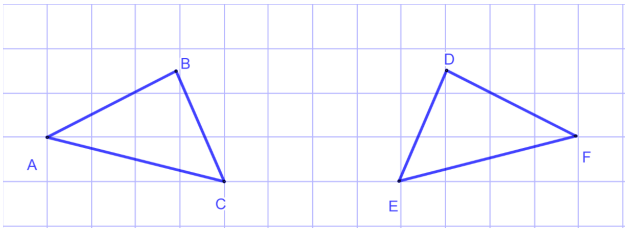
other.



8. Quadrilateral ABCD is congruent to quadrilateral A”B”C”D”. Describe a sequence of transformations that would map quadrilateral ABCD to quadrilateral A”B”C”D”.

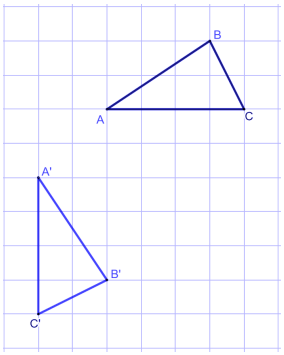


9. Are the triangles given below congruent? If yes, specify the transformation or combination of transformations that you can perform so that one quadrilateral will map onto the other quadrilateral



10. Are the triangles given below congruent? If yes, specify the transformation or combination of transformations that you can perform so that the one triangle will map

onto the other.



7. Constructions of Geometric Figures

Constructions

For a review of constructions by using a compass and straight edge, watch the videos on constructions: https://youtu.be/IrRTGbqr8MQ?t=109

GeoGebra's Tools for Construction

In GeoGebra, constructions using a compass and ruler can be mimicked by using **Move, Point, Line, Ray and Compass** tools.

Copy a Line

Watch the video: https://youtu.be/SpTLhjbaRGc?list=PL63A209B3390B97CD&t=44

Copy an Angle

Watch the video: https://youtu.be/Levf77WyZvE?list=PL63A209B3390B97CD&t=114

Exercise 7: Constructions of Geometric Figures

Make the following constructions by using **Compass, Point,** and **Line** tools and for each of these

(a) Describethe steps in the construction.

(b) Explain how you know that your construction has accomplished the task.

(c) Use the move tool to drag various elements of construction (independent as well as dependent) and state what do you notice when you move them

1. Draw a line segment and construct a copy of the line segment.
2. Draw an angle and construct a copy of the angle.
3. Draw a line segment and construct its bisector.
4. Draw an angle and construct its bisector.
5. Draw a line segment and construct a perpendicular to the line segment.
6. Draw a line and mark a point P outside it and construct a line parallel to the given line passing through point P.
7. Construct an isosceles triangle
8. Construct an equilateral triangle
9. Construct a regular hexagon
10. Construct the following quadrilaterals by using properties of their diagonals:
    (a) a rhombus
    (b) a kite
    (c) a rectangle
    (d) a square
    (e) a parallelogram

If you have difficulty with a construction watch the videos in the playlist: https://www.youtube.com/playlist?list=PL63A209B3390B97CD and then construct them. If you can't figure out how to do a construction given in Q.10, watch the video: https://youtu.be/tNCum6ujliQ?t=336 and then construct it.

8. Dilation and Proving Similarity by Transformations

Dilation is the process of resizing or transforming an object. It is a transformation that makes the objects smaller or larger by a scale factor. The new figure obtained after dilation is called the **image** and the original image is called the **preimage**.
Examples of dilation are zoom in and zoom out in computer, enlargement of photos.

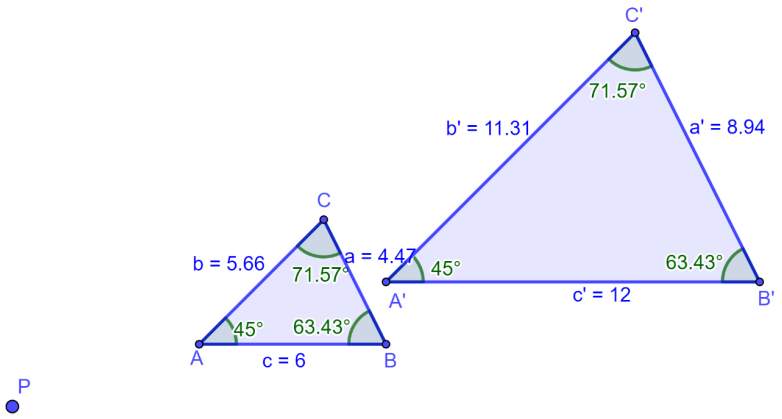
Dilation of figures

1. Draw the object to be dilated and a plot a point P called the **center of dilation** from which the object is to be dilated. It may be a vertex of the figure, its center or any other point outside the figure.
2. Create a slider **a** for the scale factor.
3. Select **Dilate** from the dropdown menu of the **Transformation** tool.
4. Select the object to be dilated and center of dilation P specifying **a** as a factor of dilation. An image with dilation factor **a** will be drawn.

Properties of Dilated Image

1. If we move the point on the slider for **a**, we will notice that the dilated image is smaller than the preimage for **a** < 1, congruent to the image for **a** = 1, and larger than the preimage for **a** > 1.
2. If we measure the length of corresponding sides of the preimage and its image after dilation and find the ratios of the corresponding sides of its image after dilation to lengths of preimage, we will notice that all the ratios are equal to the factor of dilation.
3. Measure the distance from the point P to the vertices of image and preimage. Find the ratios of the distances from the point to the corresponding vertices. We will notice that all the ratios are equal to the factor of dilation.
4. Find the measures of corresponding angles of the preimage and image. We will notice that all of them are equal.

Example 1: The figure given below shows a triangle ABC and its image A'B'C' after dilation from point P by a factor of 2.



State what do you notice

(a) about the relation between angles of the preimage and image of the triangle.

(b) about the ratio of lengths of all corresponding sides.

(c) Drag point P in different locations including at the center of the triangle, a vertex. What do you notice?

(d) Drag a vertex of the triangle. What do you notice?

Answer

(a) All angles are equal in triangle ABC and its image A'B'C' after dilation.

(b) The ratio of lengths of all corresponding sides of the image and after dilation to lengths of preimage are equal.

(c) If we move point P, the location of its image A'B'C' changes, but the corresponding angles remain equal and the ratio of length of the corresponding sides of its image after dilation to lengths of preimage are equal for all locations of P.

(d) When we drag a vertex of the triangle, the shape of the triangle changes, so also of its image However the corresponding angles and the ratio of length of the corresponding sides of the image dilation to lengths of preimage are equal.

Similar Figures

Two figures are similar($\sim$) if they have the same shape. Thus any figure and its image after dilation are similar.

If two figures are similar, all angles are equal in them and the ratio of lengths of all corresponding sides of its image after dilation to lengths of preimage are equal.

Proving Similarity of Two Figures

We can prove the similarity of two figures by trying transformation of one figure by dilation or a combination of dilation and one or more of rigid transformations to coincide with the other figure. If it coincides with it, the figures are similar, otherwise they are not similar.

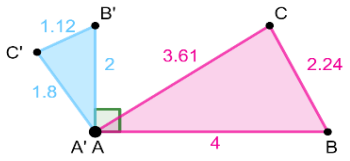
For a dynamic illustration of similarity, watch the video: https://youtu.be/dxF171zKcHY?t=21

Similar Triangles

Just like congruence for we can prove that triangles are similar, if they satisfy any of these sets of conditions:

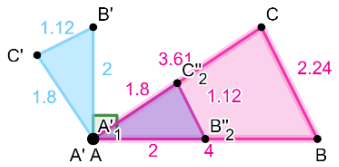
- Two angles of one triangle are congruent to two angles of another triangle(AA).
  For a dynamic illustration of AA similarity, watch the video:
  https://youtu.be/w71xEi7N1pk?t=33
- An angle of one triangle is congruent to an angle of another triangle, and sides that include the angles are proportional(SAS).
  For a dynamic illustration of SAS similarity, watch the video:
  https://youtu.be/kG1deXchc7Y?list=PLIlF_FGjAuZVZTEFth_X9G2Gvt0nFwA9L&t=36
- All corresponding sides are proportional(SSS).
  For a dynamic illustration of SSS similarity, watch the video:
  https://www.youtube.com/watch?v=Y_Dq_8NM-xc&pp=ygUdU1NTIHNpbWlsYXJpdHkgdXNpbmcgZ2VvZ2VicmE%3D

Example 2: The ratios of all corresponding sides of the triangles given below are equal for all sides(SSS).



Prove △A'B'C' ~ △ABC by transforming △A'B'C' to coincide with △ABC and describe the transformation or sequence of transformations that you used.

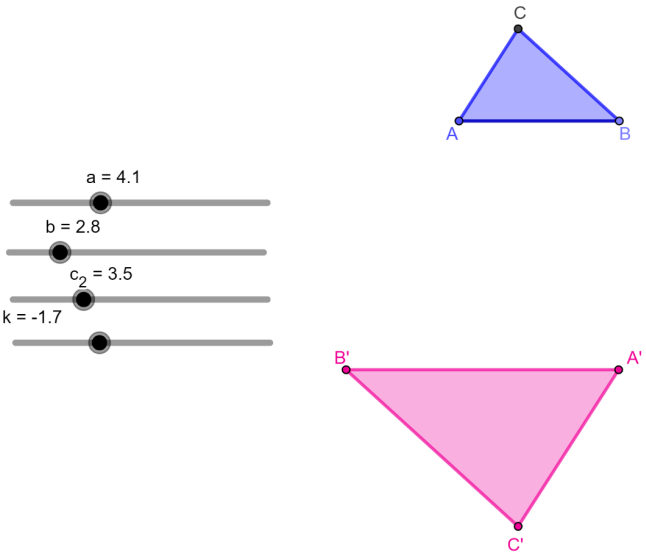
1. Rotate △A'B'C' by 90° clockwise $\triangle A_2'B_2'C_2'$ will show.
2. Dilate $\triangle A_2'B_2'C_2'$ by the ratio of any two corresponding sides that is 2, the dilated triangle coincides with △ABC and



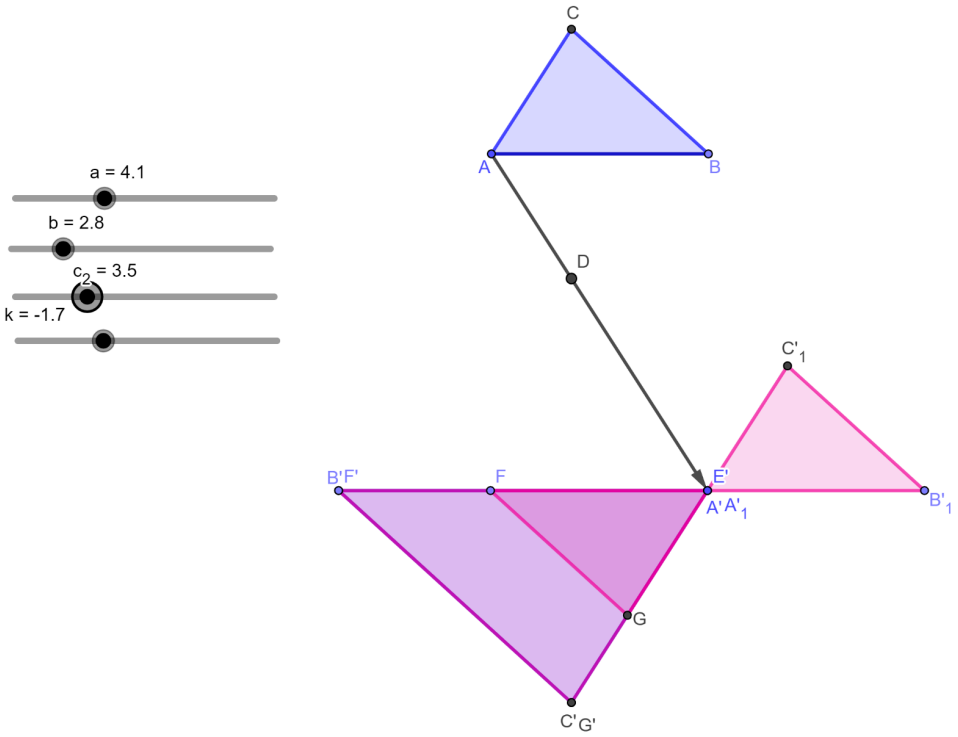
Example 3: Triangles ABC and △A'B'C' have ratios of all their corresponding sides equal to k.

(a) Prove △ABC ~ △A'B'C' by transformation of ABC to coincide with △A'B'C', and describe the transformation or sequence of transformations that you used.

(b) If you move the points on the sliders to change their values, what do you notice?



Answer

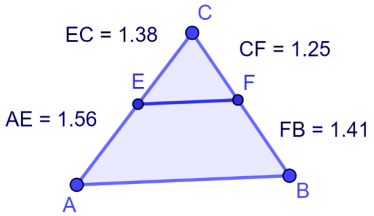


(a)

1. Draw a vector from A to A'.
2. Translate △ABC by the vector AA'. It will show $\triangle A'_1B'_1C'_1$ image of △ABC .
3. Reflect $\triangle A'_1B'_1C'_1$ about point A'. It will show △E'FG.
4. Dilate △E'FG by a factor of FG/B'C'. Its image will coincide with △DEF.

(b) If we move the points on sliders to change their values, if a triangle is possible, the preimage and final image remain coincident. It demonstrates, it holds for all such triangles.

Example 4: Verify the theorem: A line parallel to one side of a triangle divides the other two proportionally.



1. Draw a triangle ABC.
2. From any point E on $\overline{AC}$, draw a line parallel to the line segment AB.
3. Mark its points of intersection with side BC as F.

4. Measure the lengths of line segments AE, EC, BF and FC using the Distance tool, and find the ratios of lengths of $\frac{AE}{EC}\ and\ \frac{BF}{FC}$. You will notice that they are equal.
5. If you drag any vertex of the $\triangle ABC$, you will notice that although the lengths of line segments AE, EC, BF and FC, change, the ratios of lengths of $\frac{AE}{EC}\ and\ \frac{BF}{FC}$ remain equal. Thus verifying the theorem.

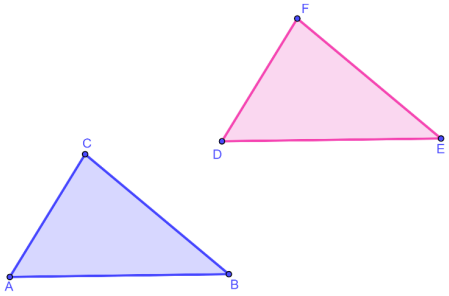
Exercise 8: Dilation and Proving Similarity by Transformations

1. Draw a triangle and dilate it by a factor of **k** from
   (a) a vertex
   (b) its center by a factor of **k**.
   (c) a point P outside it.
   For each of these
   (i)) Find all the angles of the preimage and image. What do you notice?
   (ii) Find the ratios of all the corresponding sides of the preimage and image. What do you notice?
   (iii) Find the ratios of the distances between point P and the corresponding vertices of the preimage and image. What do you notice?
   (iv) Move the point on the slider of **k**, what do you notice?
   (v) Drag a vertex of the triangle, what do you notice?
2. Draw a quadrilateral and dilate it by a factor of k with
   (a) a vertex as its center of dilation by a factor of **k**.
   (b) its center as its center of dilation by a factor of **k**.
   (c) a point P outside it as its center of dilation
   For each of these
   (i) Find all the angles of the preimage and image. What do you notice?
   (ii) Find the ratios of all the corresponding sides of the preimage and image. What do you notice?
   (iii) Find the ratios of the distances between point P and the corresponding vertices of the preimage and image. What do you notice?
   (iv) Change the values of k, what do you notice?
   (v) Drag a vertex of the triangle, what do you notice?
3. Draw a circle and dilate it by a factor of **k** from a point P outside the circle.
4. Demonstrate all line segments are similar.
5. Demonstrate that all regular polygons with the same number of vertices are similar.
6. Prove all circles are similar.

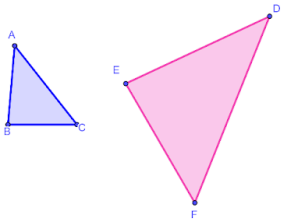
For Q. 7-10, go to https://www.geogebra.org/m/fnb7wagz and find the figures for these questions. Select the figure and click ⋮ and select **open in app**. It will open it in GeoGebra and you can do the exercise.

7. Prove $\triangle ABC \sim \triangle DEF$ by transforming $\triangle ABC$ to coincide with $\triangle DEF$ and describe the sequence of transformations that you carried out to prove the figures are similar.

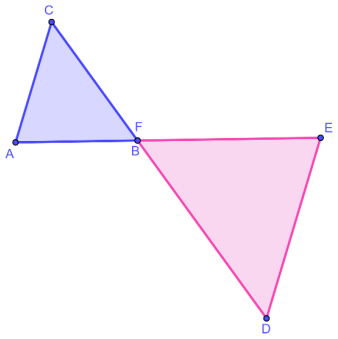
(a)



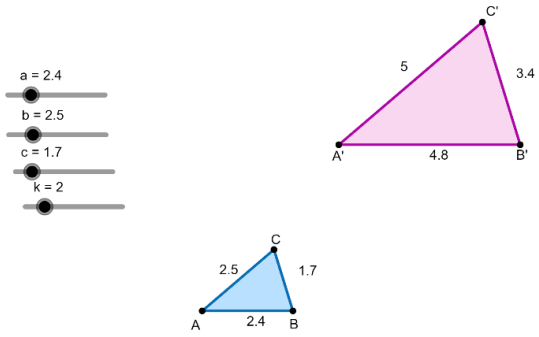
(b)



(c)



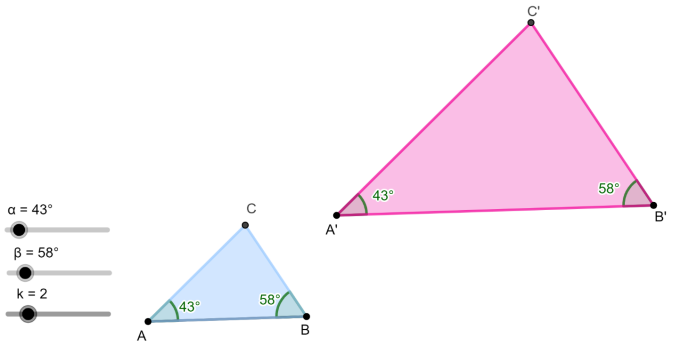
8. (a) Prove ΔA'B'C' ~ △ABC by transforming △ABC to coincide with △A'B'C' and describe the sequence of transformations that you carried out to prove the figures are similar.
   (b) Change the lengths of sides of the triangle ABC by moving the points on the sliders giving the lengths of the sides of the △ABC, what do you notice?
   (c) Change **k** the factor of dilation, what do you notice?



9. (a) Prove ΔA'B'C' ~ △ABC by transforming △ABC to coincide with △A'B'C' and describe the sequence of transformations that you carried out to prove the figures are similar.

(b) Change the angles of the triangle ABC by moving the points on the sliders of the angles of △ABC, what do you notice?

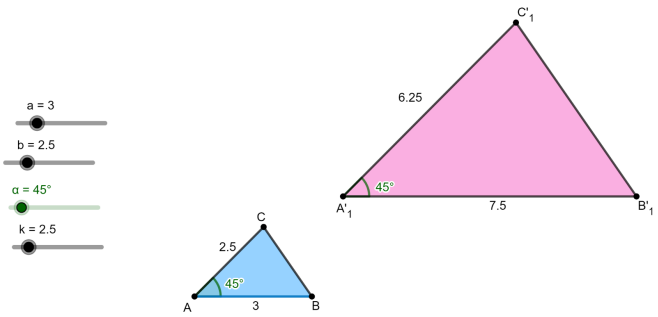
(c) Change **k** the factor of dilation, what do you notice?



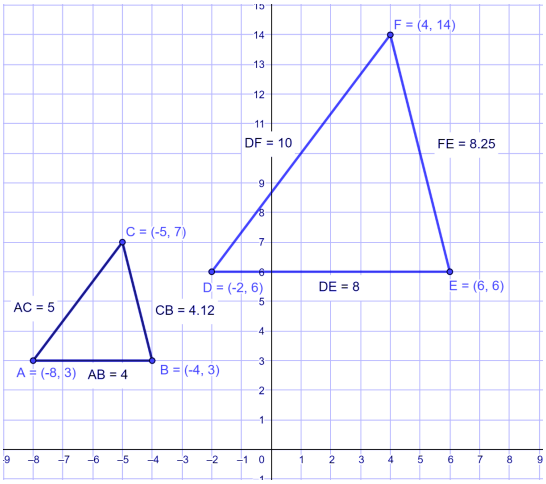
10. (a) Prove ΔA'B'C' ~ △ABC by transforming △ABC to coincide with △A'B'C' and describe the sequence of transformations that you carried out to prove the figures are congruent.

(b) Change the length of sides and angle included between them of triangle ABC by moving the points on the sliders for them, what do you notice?

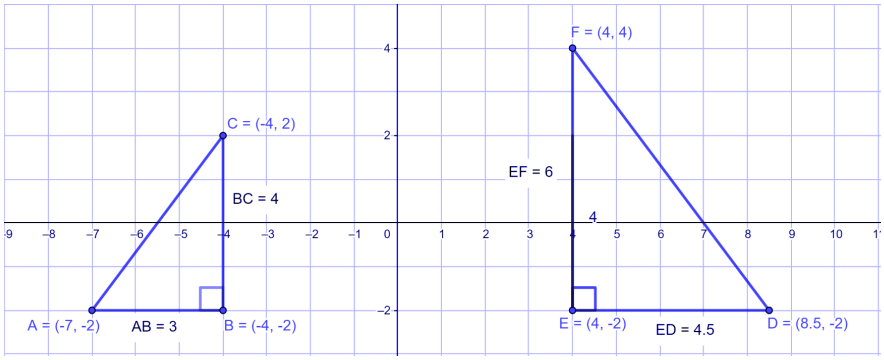
(c) Change **k** the factor of dilation, what do you notice?



11. Is △ ABC ~ △DEF? If yes, describe a sequence of transformations that you may carry out to show that the figures are similar.

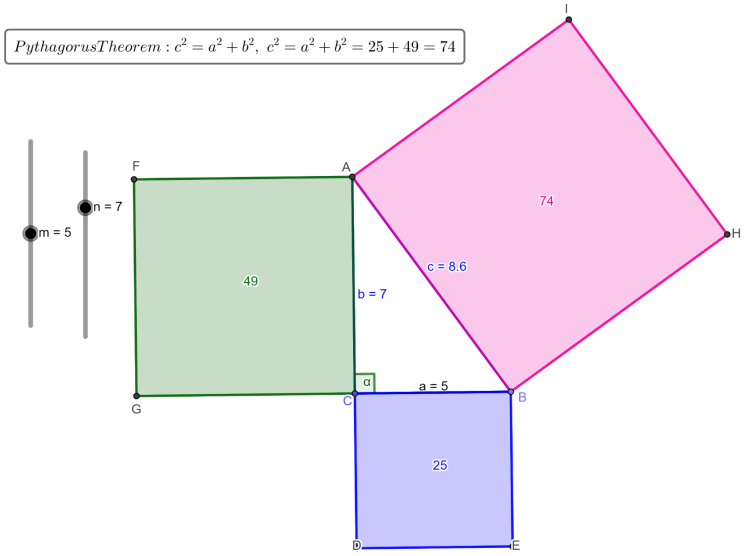


12. Is △ ABC ~ △DEF? If yes, describe a sequence of transformations that you may carry out to show that the figures are similar.



9. Trigonometric Ratios

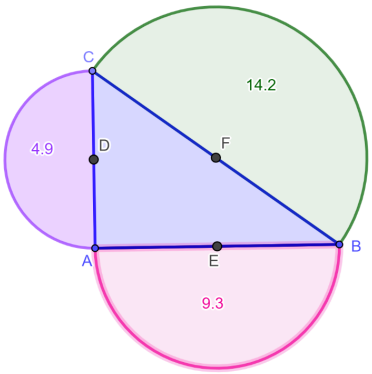
Demonstrate Pythagoras Theorem



1. Activate the Slider tool and create sliders for **m** and **n**, the lengths of the sides a and b of the triangle.
2. Draw $\overline{CB}$ of length **m**.
3. Draw a line perpendicular to $\overline{CB}$ at point C.
4. With C as center and radius **n**, draw a circle and mark the point where the circle and perpendicular bisector intersect as A.
5. Draw a right triangle ABC by activating the **Polygon Tool** and selecting points A, B, and C and A again to draw right △ABC.
6. Draw squares on sides AB, BC, and CA by activating the polygon tool and selecting points A & B; B & C; and A & C respectively and typing 4 for the number of vertices.(If a square is drawn in the opposite direction, then select the points in opposite order).
7. Calculate c^2 by typing it in the **Algebra View.**
8. Calculate $a^2 + b^2$ by typing it in the **Algebra View.**
9. You will notice that $c^2 = a^2 + b^2$.
10. If you move the points on the sliders for m and n, the lengths of the sides of the triangle will change; but relation: $c^2 = a^2 + b^2$ will hold holds for all such triangles.

For another demonstration of Pythagoras Theorem, Watch the video:
https://youtu.be/p5GnLUMiCPQ?t=21

Example 1: Draw a semicircle on the sides of a right ΔABC, and check if the sum of areas of the figure on legs is equal to the area of the figure on the hypotenuse holds for them? What do you wonder?



1. Draw a right triangle by following the instructions 1-5 given in demonstration of Pythagorus theorem.
2. Find the midpoints D, E, and F of the sides AC, AB, and BC.
3. Activate the **Circle** tool and select **Circular Sector** from its dropdown menu and select the points E,A,B; D,C,A; and F,B,C to draw semicircular sectors on them.
4. Activate the **Measure** tool and select **Area** from its dropdown menu and click inside the semicircles. It will name the semicircles as h, d, e.
5. Type d + h in **Algebra View** and you will notice that this sum is equal to e the area of semicircle on A & B. Thus the relation holds for areas of semicircles too.
6. For what other figures the sum of the areas on the sides may equal the area on the hypotenuse?

Trigonometric Ratios

For a review of Trigonometric Ratios, watch the video: https://youtu.be/q5JMy_NAP1c?t=8
You can find the value of any trigonometric ratio by typing it in the **Algebra View**.

The Law of Sines

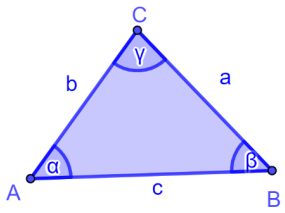
For a △ ABC, the law of sines is given by

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

It can be used to solve a triangle i.e. find lengths of remaining sides and angles, if you are given: measures of two angles and length of a side(ASA or AAS) or lengths of two sides and measure of a non-included angle(SSA).
For how to do it watch the video: https://youtu.be/FPVGb-yWj3s?t=6

Demonstrate the law of sines.

$\frac{sin(\alpha)}{a} = \frac{sin(\beta)}{b} = \frac{sin(\gamma)}{C}$.



1. Draw a triangle ABC.
2. Activate the **Measure** tool and select Angle from its dropdown menu and click inside the triangle. It will label angles A, B and C as $\alpha,\ \beta,\ and\ \gamma$ respectively.
3. Type in the **Algebra View** sinα/a. It will give the value of the ratio.
4. Type in the **Algebra View** sinβ/b. It will give the value of the ratio.
5. Type in the **Algebra View** sinγ/c. It will give the value of the ratio.
6. You will notice that $\frac{sin(\alpha)}{a} = \frac{sin(\beta)}{b} = \frac{sin(\gamma)}{C}$.
7. If you move any of the points A, B, or C, you will notice that while the values of a, b, and c and angles $\alpha,\ \beta,\ and\ \gamma$ will change but the relation $\frac{sin(\alpha)}{a} = \frac{sin(\beta)}{b} = \frac{sin(\gamma)}{C}$ will still hold.

The Law of Cosines

For a △ ABC, the law of cosines is given by

$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot cosA$

$b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot cosB$

$c = a^2 + b^2 - 2 \cdot a \cdot b \cdot cosC$

It can be used to solve a triangle that is find lengths of remaining sides and angles, if you are given:

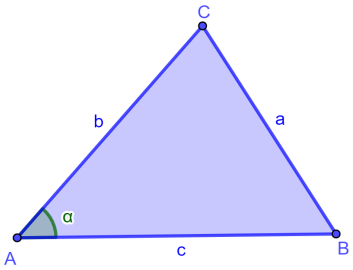
Lengths of three sides(SSS)

Lengths of two sides and measure of the included angle(SAS).

For how to do it watch the video: https://youtu.be/oZh19gjZyMI?t=33

Demonstrate the law of cosines given two sides and an included angle.

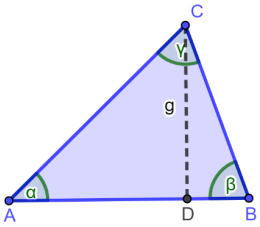
Demonstrate $a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot cos(\alpha)$



1. Create a slider for □α.
2. Draw a ΔABC with length of sides b and c and m∠A = α.
3. Calculate a^2 by typing it in the **Algebra View.**
4. Calculate $b^2 + c^2 - 2 \cdot b \cdot c \cdot cos(\alpha)$ by typing it in the **Algebra View.**
5. You will notice that $a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot cos(\alpha)$.
6. If we drag point C, it changes the length of sides b, c, and α, $but\ a^2$ will remain equal to $b^2 + c^2 - 2 \cdot b \cdot c \cdot cos(\alpha)$.

Similarly we can show $b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot cos(\beta)$ and $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot cos(\gamma)$.

Example 2: Find all angles given three sides of a triangle



1. Draw a triangle ABC whose sides have lengths a, b, and c. Activate the **Angle** tool and click inside the triangle. It will name the angles as α, β, and γ.
2. Draw a perpendicular line from point C to line segment AB
3. Label the point of intersection as D and hide the line.
4. Draw $\overline{CD}$ and label it g.
5. Then sin(β) = g/a. the **Algebra View** will show the value of the angle β.
6. After we have found β, we can find the angles α and γ by using the law of sines..

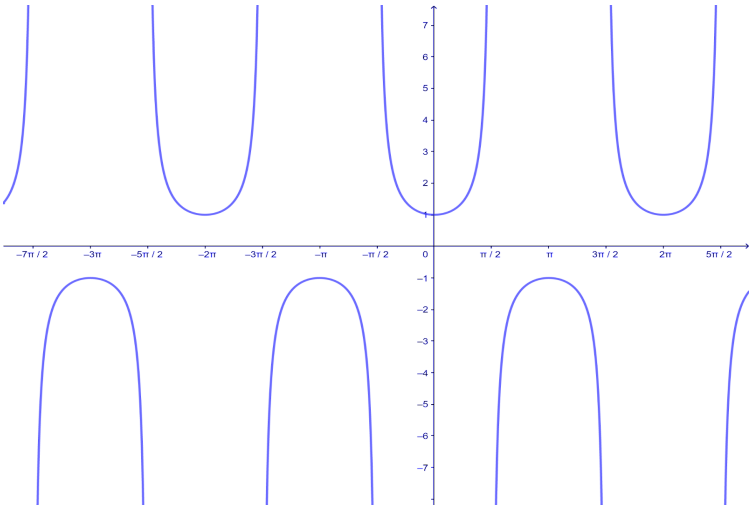
Trigonometric Functions

Trigonometric functions are real functions which relate an angle of a right triangle to ratios of two side lengths. Each of these six trigonometric functions has a corresponding inverse function.

We can draw the graphs of trigonometric functions by typing the function in the **Algebra View**.

If we want to label the angles in terms of π, we can change the distance on x-axis to $\frac{\pi}{2}$ by clicking , and selecting and then x-axis and change the distance between units to $\frac{\pi}{2}$ and y-axis to 1 by selecting x-axis and y-axis.

Example 3: Draw the graph of sec(x).

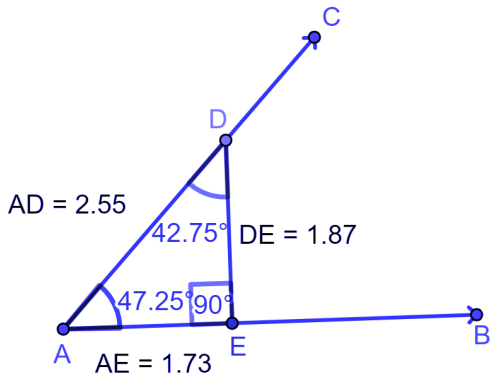


Inverse trigonometric functions

Inverse trigonometric ratios can be used to find the value of the unknown angle if we know the measure of the ratio of two sides of the right-angled triangle. For example if sin(θ) = (Opposite)/(Hypotenuse), then θ = $\sin^{-1}$(Opposite)/(Hypotenuse). Inverse trigonometric ratios are also denoted by arcsin(θ).
We can find these by typing it in the **Algebra View** or using a calculator.
Similarly inverse of other trigonometric ratios may be defined and found.

Exercise 9: Trigonometric Ratios

1. Construct the following figures on the sides of a right ΔABC, and check if the sum of areas of the figure on legs is equal to the area of the figure on the hypotenuse holds for them?
   (a) An equilateral triangle.
   (b) A regular hexagon.
2. (a) Draw an □BAC.



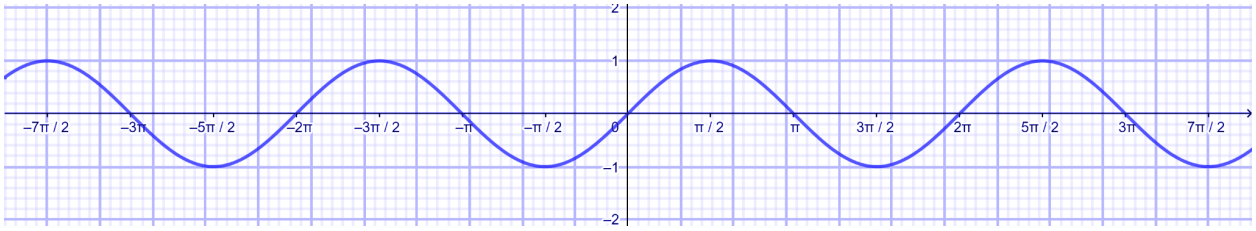
   (b) Mark a point D on ray AC. Construct a line perpendicular to ray AB and label its point of intersection with ray AB as E.
   (c) Measure □A and lengths of sides DE, AE and AD of △AED.
   (d) Calculate the ratios $\frac{DE}{AD}$, $\frac{AE}{ED}$, and $\frac{DE}{AE}$. by typing them in the Algebra View. Name the trigonometric ratios they represent.
   (e) Drag D along AC, what happens to the measure of ∠A if you drag point D?
   (f) As you move D, what happens to the values of the three ratios you calculated? (You can check that in the **Algebra View**)
   (h) Explain this result.
3. Find the values of Trigonometric ratios given below using GeoGebra or a calculator.
   (a) sin 40°
   (b) cos 13°
   (c) tan 32°

4. .(a) Complete the table given below for a △ABC right angled at C

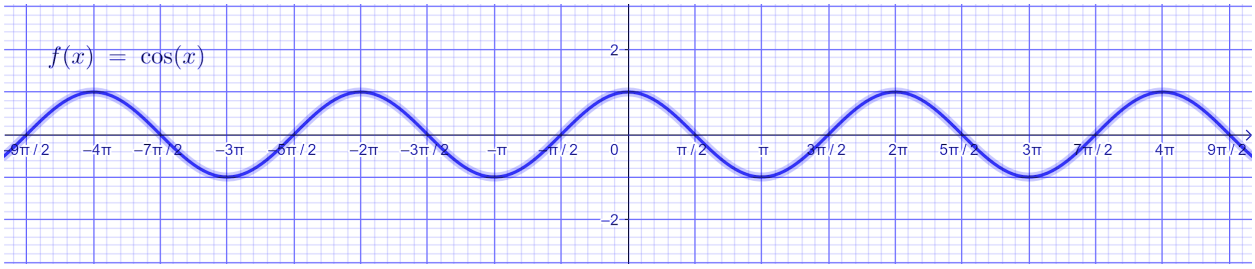
Trigonometric ratio	$\angle A = 30^\circ$	$\angle A = 45^\circ$	$\angle A = 60^\circ$
sin A			
cos A			
tan A			
sin B			
cos B			
tan B			

(b) What type of angles are A and B?
(c) What relations do you notice between trigonometric ratios of ∠A and ∠B.
(d) Would this relation hold for any values of the two acute angles in a right triangle?

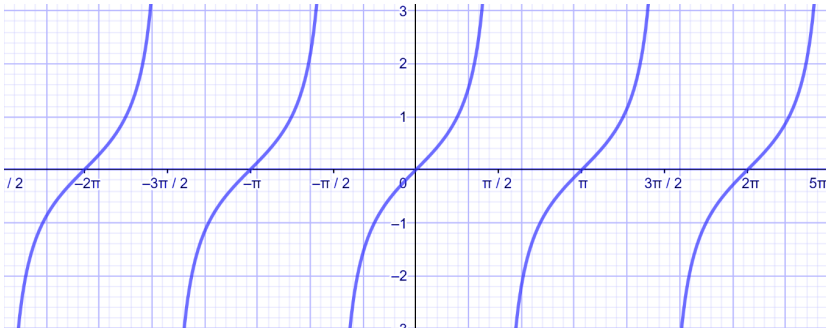
5. If sin 40° = 0.64, find cos 50°.
6. If tan 60° = 1.73, find cot 30°.
7. The graph of sin(x) is given below, Find
   (a) sin 0, sin$\frac{\pi}{2}$, and sin(π)
   (b) maximum and minimum values that sin(x) can have.



8. The graph of cos(x) is given below, find
   (a) cos 0, cos$\frac{\pi}{2}$, and cosπ.
   (b) maximum and minimum values that cos(x) can have



9. The graph of tan(x) is given below, find
   (a) tan 0, tan $\frac{\pi}{2}$, and tan π.
   (b) the maximum and minimum values tan(x) can have.

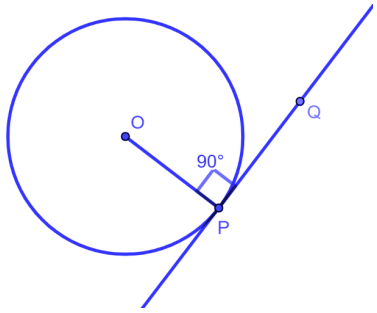


10. Draw a triangle and verify the law of sines.
11. Draw a triangle and verify the law of cosines.
12. Draw the graphs of the following trigonometric ratios.
    (a) sin(x)
    (b) cos(x)
    (c) tan(x)
    (d) cosec(x)
    (e) sec(x)
    (f) cot(x)
13. Interpret the graphs(e.g. their domain, range, behavior, maximum and minimum values) drawn in Q.4.
14. Find the following angles in degrees:
    (a) $sin^{-1}(0.5)$.
    (b) $cos^{-1}(0.6)$
    (c) $tan^{-1}(-1)$

10. Demonstration of Theorems Relating to Circles

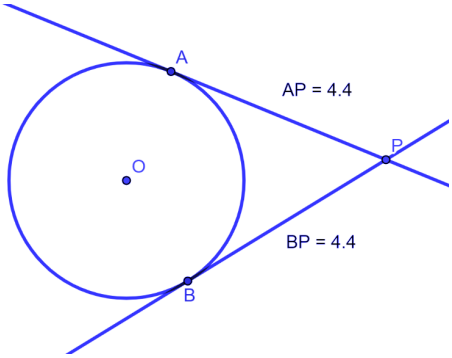
We can explore relations between angles, radius, chords, secants, and tangents to a circle by drawing and measuring them and using the drag tool.

If a line is a tangent to the circle, then the line is perpendicular to the radius at a point of tangency



1. Draw a circle with center O. Plot a point P on the circle.
2. Activate **tool** and select the **Tangents** from the dropdown menu of the tool. Select the point P and the circle. A line touching the circle called a **tangent** will be drawn.
3. Draw radius of the circle using the Segment tool. Plot a point Q on the tangent and measure the □QPO. We notice that it is a right angle. Thus a tangent is perpendicular to the radius at the point of tangency.
4. If we drag the point P by using the **Move** tool. We will notice that the tangent at any point remains perpendicular to the radius.

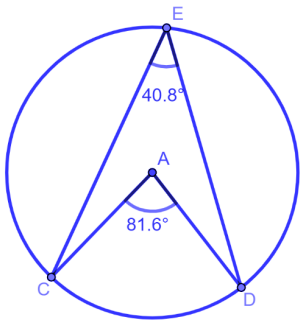
The two line segments that are tangent to a circle from a point outside it are congruent.



1. Draw a circle. Plot a point P outside the circle.
2. Activate **tool and** select the **Tangents** from the dropdown menu of the tool. Select the point and the circle. Two lines touching the circle called **tangents** will be drawn.

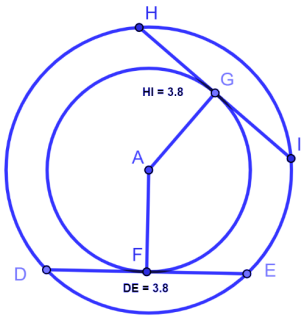
3. Label the points of intersection of the tangents with the circle as A and B.
4. Measure the distance between P and A and P and B. We will notice that PA = PB.
5. Drag point P by using the **Move** tool. We will notice that PA remains equal to PB. Thus two line segments that are tangent to a circle from a point outside it are congruent.

Measure of an angle subtended by any arc at the center is twice that inscribed by it.



1. Draw a circle with center A. and plot points C and D on the circle to mark arc CD of the circle. Plot another point E on the circle and draw central □CAD and inscribed □CED.
2. Measure □CAD. Compare it with □CED. We will notice that measure of □CAD is twice the measure of □CED.
3. Drag the point E by using the **Move** tool.We will notice that the relation between the measures of the central angle and inscribed angle by the same arc holds for any point E on the circle.

Demonstrate Chords equidistant from the center of a circle are equal.



1. Draw a circle c with center A and any radius.
2. Draw a chord DE in it.
3. Draw a perpendicular bisector of chord DE.
4. Mark the point of intersection of Chord DE and perpendicular bisector as F.
5. With center A and radius AF draw a circle.

6. Mark a point G on the circle and draw a tangent at point G. Mark its points of intersection with circle c as H and I.
7. Measure the lengths of chords DE and HI. We will notice that they are equal.
8. Drag the point G, you will notice that chord HE moves with it and its length always remains equal to the length of the chord CD. It demonstrates that all chords equidistant from the center are equal.

Exercise 10: Demonstration of Theorems Relating to Circles

1. Draw a circle passing through a given point.
2. Draw a circle with radius of 3 units.
3. Draw a circle and draw in it a
   (a) chord.
   (b) radius.
   (c) diameter.
   (d) secant.
4. Demonstrate that measures of all angles Inscribed by a chord by joining the end points of a chord to a point on the circle are equal
5. Demonstrate that angle subtended by a diameter at any point on the circle is a right angle.
6. Demonstrate that a perpendicular drawn from the center of a circle to a chord bisects the chord.
7. Demonstrate a line joining the center of a circle to the midpoint of a chord is perpendicular to the chord.
8. Demonstrate the sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
9. Demonstrate all circles are similar.
10. Draw a triangle and inscribe a circle in it.
11. Draw a triangle and circumscribe a circle passing through its vertices.
12. How will you locate the center of the circle? Describe your procedure.

11. Coordinate Geometry

GeoGebra Commands

So far we have been mainly using the toolbar to draw figures, demonstrate their properties, prove and verify theorems.

We can also type commands in the **Algebra View** or **Input Bar** to draw objects and objects related to them and measure their attributes.

When we type a command in the **Algebra View**, and press **enter**. The geometrical object e.g. a point, a line, or a curve are automatically labeled and the attributes of the object e.g. the coordinates of the point, equation of the line or curve, the length of a line segment or arc, area of polygon, area of a circle or sector of a circle are shown in the **Algebra View** and the objects are drawn in the **Graphic View.**

Examples of Commands

The following commands may be used for drawing or measuring attributes of an object, by typing the command and pressing **Enter.** These objects are then drawn in the **Graphic view** and automatically labeled.Some attributes of these are given in the **Algebra View**.. These labels may then be used for finding other attributes of objects and their measures.

Plotting a point with coordinates (x,y): (x,y)
Drawing a line: Line(A,B)
Drawing a segment: Segment(A,B)
Drawing a segment with given length a starting at point A:Segment(A,a)
Midpoint of a line segment: Midpoint(A,B)
Distance between two points: Distance(A,B)
Point of intersection of two lines with labels g and h: Intersect[g,h]
Slope of a line labeled f: Slope(f)
Line through a point P parallel to the line labeled f: Line(P,f)
Reflect a polygon labeled poly1 about a line f: Reflect(poly1,f)
Convert a decimal number 0.4 into a fraction: FractionText(0.4)

For all commands in Geometry see https://wiki.geogebra.org/en/Geometry_Commands

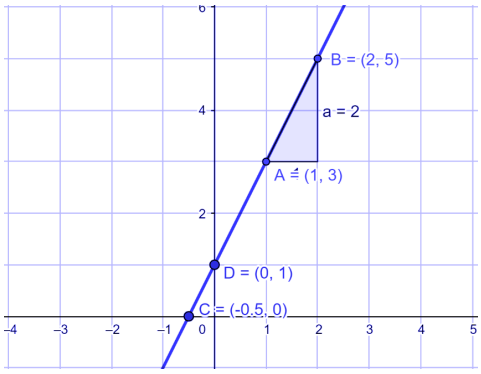
Draw a line with a given equation.

We can draw a line in the **Graphic View** by typing its equation in any form(slope-intercept, standard form or a line passing through a point (x_1, y_1) and having a slope m e.g. y = 2x + 3, 3x + 4y = 12 or y - y_1 = m(x - x_1) in the **Algebra View.**

Draw graphs of Functions

We can draw a graph of any function in the graphic view by typing its equation in the **Algebra View**

Draw a line passing through two points A = (1,3) and B = (2,5)



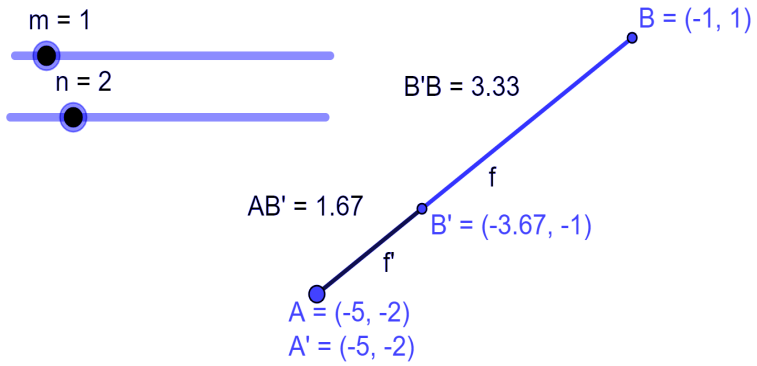
Type in the input bar

1. A = (1,3). It will plot point A in the **Graphic View**.
2. B = (2,5). It will plot point B in the **Graphic View**.
3. Line(A,B). It will show f = Line(A,B) = $2x - y = -1$ in the **Algebra View** and a line passing through A and B will be drawn in the **Graphic View**.

We can change the equation into slope intercept form by selecting the line, right clicking it, and selecting **y = mx + b**.

We can find x-intercept and y-intercept of the line by clicking ⋮ in the equation of line and selecting **Special Points** there. The values of these will be given in the **Algebra View** and plotted in the **Graphic view**.

Example 1: **Find a point and on a line segment that divides it in the ratio m:n**



1. Plot two points A and B.
2. Draw line segment $\overline{AB}$.
3. Create sliders for integers m and n
4. Type ***l*** = m + n in the Algebra view.
5. Dilate $\overline{AB}$ by a factor $\frac{m}{l}$ from point A. A point A' coincident with A and B' with its coordinates will be shown on $\overline{AB}$. The point B' divides the line segment AB in the ratio m:n.
6. We can divide the line in other ratios by changing the values of m and n by moving the points on their sliders.

We can verify it by measuring the lengths of A'B' and B'B by Distance command or tool and finding their ratio which should equal $\frac{m}{n}$.

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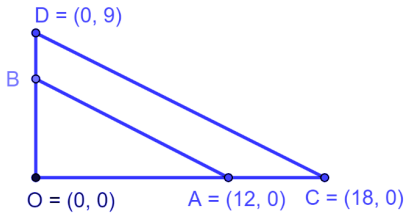
Exercise 11: Coordinate Geometry

1. Plot the following points.
   (a) (2,4).
   (b) (3, -4).
   (c) (-1, 6).
   (d) (-5, -5).
2. Draw lines whose equations are given below:
   (a) y = 2x - 3
   (b) y - 3 = 2(x - 4)
   (c) 3x - 2y + 12 = 0
3. Draw a line whose equation is y = mx + b.
   (a) Move the point on the slider for m, what do you notice?
   (b) Move the point on the slider for b, what do you notice?
4. Draw a circles whose equations is given by $(x - h)^2 + (y - k)^2 = r^2$. what do you notice as you move the points on sliders of
   (a) h.
   (b) k.
   (c) r.
5. Draw the graphs of the function. $y = a(x - h)^2 + k$. What do you notice as you move the point on the slider of
   (a) h.
   (b) k.
   (c) a.
6. Find the distance between points whose coordinates are
   (a) (2,4) and (5,4).
   (b) (3,5) and (5,3).
   (c) (2, -4) and (6, -3).
7. Find the midpoint of a line segment the coordinates of whose endpoints are
   (a) (2,5) and (4,6).
   (b) (2,-4) and (6,-2).
8. Three vertices of a parallelogram are A = (1,-2), B = (-2,3) and D = (5,-1), Find the coordinates of the fourth vertex.
9. If A = (-3,2), B = (-2,7), C = (2,4) and D = (1,-1), Show that ABCD is a parallelogram.

10. △AOB ~ △COD. Find the coordinates of point B and the scale factor.



11. Find the reflection of point A = (3,5) over the following points or lines and state what do you notice about (i) the coordinates of A and its image A' and (ii) about the distance from point A and A' to the line of reflection?
    (a) x axis.
    (b) y axis.
    (c) origin.
    (d) the line y = x.
12. Find the reflection of point A = (a,b) over the line y = x and find the coordinates of its image A'. What relation do you notice between the coordinates of point A and coordinates of its image A' as you move the point on sliders of a and b.
13. Draw a circle with radius 3 and the center as
    (a) origin.
    (b) point (2,-4).
14. (a) Draw a line parallel to y = 2x + 3 and passing through a point P (-5,2).
    (b) Write its equation.
15. (a) Draw a line perpendicular to y = $\frac{1}{3}$ x + 5 and passing through the point (4,-1).
    (b) Write its equation.
16. (a) Draw a line whose equation is y = mx + b.
    (b) Draw a line parallel to it.
    (c) Find the slope of both the lines.
    (d) Move the point on the slider for m, what do you notice about the relation between the slopes?
    (e) Move the point on the slider for b, what do you notice about the relation between the slopes?
17. (a) Draw a line whose equation is y = mx + b.
    (b) Draw a line perpendicular to it.
    (c) Find the slope of both the lines and product of the slopes
    (d) Move the point on the slider for m, what do you notice about the relation between the slopes?
    (e) Move the point on the slider for b, what do you notice about the relation between the slopes?
18. Find a point on a line segment between two points that partitions the segment in a ratio of 2:3.

12. Perimeters and Areas of 2D figures

For a review of perimeters of 2D figures watch the videos given below:

Polygons: https://youtu.be/CA5O69Fv9NY?t=112
Circles: https://youtube.com/shorts/bH-ftsPWga4?si=iJ8-OAIEMD8jn9LGiJ8-OAIEMD8jn9LG

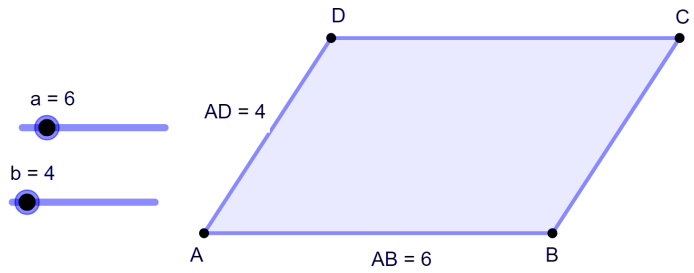
For a review of areas of 2D figures watch the videos given below:

Quadrilaterals: https://youtu.be/rB9VrzCDHVA?t=304
Circles: https://youtu.be/YokKp3pwVFc?t=143

For a review of length of an arc and area of a sectors of a circle watch the video:
https://youtu.be/Wcv0f5PpTv0?t=303

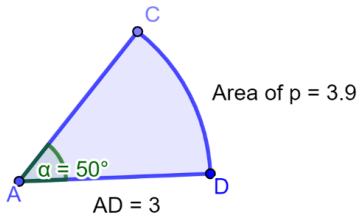
In GeoGebra the area of a polygon or circle can be found by selecting **Area** from the dropdown menu of the **Measure** tool. It is also given in the **Algebraic View**.

Example 1: Draw a parallelogram whose perimeter is 20 cm and angle between adjacent sides is 40° Can you draw many shapes with these perimeters? If yes, explain how.



1. Create sliders for a and b the lengths of sides of a parallelogram.
2. Draw a line segment AB = a cm.
3. Draw a line segment AD = b cm. (It will be drawn coincident with line segment AB, you can move it to any position.)
4. Draw a line parallel to AB and passing through point D.
5. Draw a line parallel to AD and passing through point B.
6. Label the point of intersection of the lines parallel to AB and AD as C.
7. Draw $\overline{BC}$ and $\overline{CD}$.Hide the lines passing through B,C and C,D.
8. Move the points on sliders of a and b so that their sum a + b is equal to 10 cm. All such parallelograms will have a perimeter of 20 cm.
9. Yes, we can draw many parallelograms by moving the points on sliders of a and b, so that a + b = 10 cm.

Example 2: Draw a sector of a circle with a radius of 3 cm and a central angle of 50° and find its area.



1. Draw an $\angle 50^\circ$ at A by activating the **Angle with Given Size tool** and typing 50° for angle.
2. Draw a circle with center A and radius 3, by activating the **Circle with Given Radius** and typing 3 for the radius,
3. Label its points of intersection with sides of the angle as C and D.
4. Delete the rays forming the angle and circle by activating the **Show/Hide Object** tool.
5. Activate the **Circular Sector** tool in the dropdown menu of **Circle** tool and select Center A and points C and D. A sector with central angle α and radius r will be drawn.
6. If you activate the **Area** tool from the Measurement tool and select the sector, the area of the sector will be shown in the sector. The area of the sector of the circle is also given in **Algebra View**.

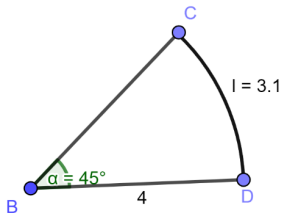
Area of a sector whose radius is 3 cm and angle at the center is 50° is 3.9 cm^2.

We can verify it by using the formula:

The area of the sector = $\frac{central\ angle}{360^\circ} \cdot \pi \cdot radius^2$.

Example 3: Draw an arc of a circle with a radius of 3 cm and a central angle of 50° and find its area.

We can draw an arc of a circle with a radius of 4 cm and a central angle of 45°. .



1. Draw an $\angle 45^\circ$ at A by activating the **Angle with Given Size tool** and typing 45° for angle.

2. Draw a circle with center A and radius 4, by activating the **Circle with Given Radius** and typing 4 for the radius,
3. Label its points of intersection with sides of the angle as C and D.
4. Delete the rays forming the angle and circle by activating the **Show/Hide Object** tool.
5. Activate the **Circular Arc** tool in the dropdown menu of **Circle** tool and select Center A and points C and D. An arc with central angle 45° and radius 4 will be drawn.
6. If you activate the **Distance** tool from the Measurement tool and select the arc, the length of the arc will be shown. The length of the arc is also given in **Algebra View**.

Length of the arc of the circle whose radius is 4 cm and angle at the center is 45 is 3.1cm^2.

We can verify it by using the formula:

The length of the arc = $\frac{central\ angle}{360^{\circ}} \cdot 2\pi \cdot radius$.

Exercise 12: Perimeters and Areas of 2D figures

1. Fill in the blanks:
   (a) 1 m = ...cm
   (b) 1 km = …m
   (c) 1 cm = …mm
   (d) 1 km^2 = ...m^2
2. Fill in the blanks:
   (a) 1 yd = ...ft
   (b) 1 ft = ...in
   (c) 1 yd^2 = ...ft^2
   (d) 1 ft^2 = ...in^2
3. A circle has a radius of 3 cm, find its
   (a) area.
   (b) circumference.
4. Interact with the app given below and answer the questions given there.
   https://www.geogebra.org/m/WFbyhq9d
5. Draw a circle whose area is 9 cm^2.
6. Find the radius of the circle whose area is 4π in^2.
7. Draw sectors of a circle whose radius and central angles are given below and find the areas of the sectors.
   (a) a central angle of 60° and radius 3 cm.
   (b) a central angle of 120° and radius 4 cm.
   (c) a central angle of 270° and radius 5 cm.
8. Draw arcs of a circle whose radius and central angles are given below and find the arc lengths of the arc.
   (a) a central angle of 60° and radius 3 cm.
   (b) a central angle of 120° and radius 4 cm.
   (c) a central angle of 270° and radius 5 cm.
9. Draw a sector of a circle with an area of
   (a) π.
   (b) 2π.
   (c) 4π.
   (d) Can you draw many sectors with these areas? If yes, explain how.
10. (a) Draw a square whose side is 5 cm. Write steps of construction.
    (b) Find its area and verify it by using the GeoGebra app.
11. (a) Construct a rectangle whose sides are 4 cm and 6 cm. Write steps of construction.
    (b) Find its area and verify it by using the GeoGebra app.
12. (a) Draw a triangle whose base is 4 cm and height 3 cm. Write steps of construction.
    (b) Find its area and verify it by using its formula.

13. (a) Draw a rhombus whose diagonals are 8 cm and 6 cm. Write steps of construction.
    (b) Find its area and verify it by using its formula.
14. (a) Draw a kite whose diagonals are 8 cm and 6 cm. Write steps of construction.
    (b) Find its area and verify it by using its formula.
15. Find the base of a triangle whose area is 12 cm and altitude is 4 cm.
16. (a) Construct a trapezoid with the lengths of parallel sides 5 cm and 4 cm and height 6 cm.
    (b) Find the area of the trapezoid and verify it by using its formula.
17. Draw shapes with perimeters given below. Can you draw many such shapes with these perimeters? If yes, describe how you will draw them.
    (a) A rectangle with a perimeter equal to 12 cm.
    (b) A rhombus with a perimeter equal to 20 cm.
    (c) A trapezoid with a perimeter equal to 24 cm.
    (d) A parallelogram with a perimeter equal to 12 cm.
18. Draw shapes with areas given below. Can you draw many such shapes with these areas? If yes, describe how you will draw them..
    (a) A rectangle with an area equal to 12 cm^2.
    (b) A rhombus with an area equal to 20 cm^2.
    (c) A trapezoid with an area equal to 24 cm^2.
    (d) A parallelogram with an area equal to 18 cm^2.
    (e) A kite with an area equal to 12 cm.
16. The perimeters of two squares are in the ratio 2 : 3, find the ratio of their areas.
17. Find the length of the other base of a trapezoid if its area = 150 cm^2, height = 10 cm and one base = 7.

13. 3D Geometry

3D Geometry is the study of 3D objects (that occupy space) and their properties.
For the name and vocabulary for describing 3D objects watch the video:
https://youtu.be/u3EcGQNdMnk?t=220

A **polyhedron** is a solid that is bounded by polygons, called faces, that enclose a single region of space. An edge of a polyhedron is a line segment formed by the intersection of two faces. A vertex of a polyhedron is a point where three or more edges meet. The plural of polyhedron is polyhedra.
A **prism** is a polyhedron with two congruent faces, called bases, that lie in parallel planes. The other faces, called lateral faces, are parallelograms formed by connecting the corresponding vertices of the bases. The segments connecting these vertices are lateral edges. Prisms are classified by the shapes of their bases.
A **pyramid** is a polyhedron whose lateral faces meet in a point. Pyramids are classified by the shape of their base.

GeoGebra can be used to study 3D Geometry. It has its own toolbar. You can activate its toolbar, by clicking ☰ at the top and selecting **Perspectives** and then clicking **3D Graphics.** The toolbar given below will appear with an **Algebra View** and a **Graphics View** that can be used for exploring 3D Geometry.

This **Tool Bar** is similar to the Geometry toolbar and has some additional tools for 3D figures.

For an explanation of the toolbar watch the video:
https://youtu.be/o6j0QkLkStk?list=PLIlF_FGjAuZVZTEFth_X9G2Gvt0nFwA9L&t=12

We can add titles for our drawings by creating a **Text Box**. For how to create the **Text Box**, watch the video: https://youtu.be/ZOugAfg37b4

Plotting points in 3D

In GeoGebra we can plot a point by writing its coordinates within parentheses.(4,2,3) in **Algebra View.** It will automatically name it as A and plot point A whose x-coordinate is 4, y-coordinate is 2, and z-coordinate is 3 in the **Graphics View**.
How to plot it on a paper, watch the video:
https://youtu.be/EzJP9uwV3ms?list=PLIlF_FGjAuZV_GYPFthdZ77StiFM_PSQw&t=6

Drawing a Plane

We can draw a plane by activating tool and selecting three noncollinear points.
We can also draw a plane by writing its equation ax + by + cz = d in the **Algebra View**.

Drawing a Plane parallel or perpendicular to a plane

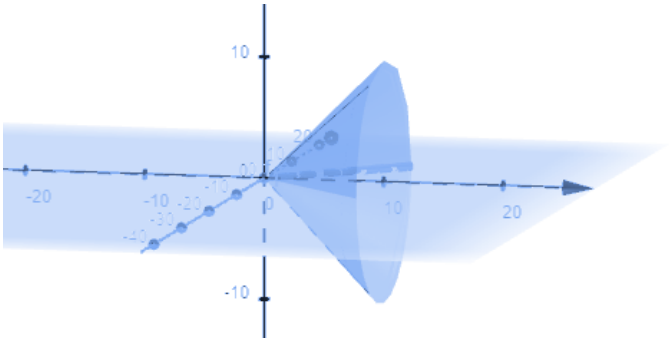
We can draw a plane parallel/perpendicular to a given plane by activating tool and selecting **Parallel/ Perpendicular Plane** from its dropdown menu, and then selecting a point/perpendicular line to the first plane.

Draw 3D shapes

We can draw 3D shapes by activating and selecting a shape from its dropdown menu and following the directions in the help that appears.

We can also draw a cone and cylinder by rotating the graph of a line about an axis in GeoGebra.

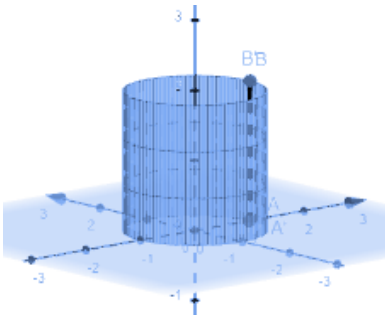
Draw a cone by rotation of a line



In the **Algebra View** type

1. k. A slider for k will be created. You can change its range and increment by selecting the slider, right clicking it and choosing settings and then slider.
2. f(x) = x, 0 < x < k. It will show a line with length k(slant height of the cone).
3. Type 0deg a slider for angle α will be created.
4. Rotate(f,n,xAxis). An image of the line will be created that will rotate as we move the point on the slider for n.
5. Surface(f,α,xAxis) as you move the point on the slider of α the surface of the cone will begin to form, when you move the point to 360°. A cone will be drawn.
6. You can change the slant height by changing the values of k.

Draw a cylinder by rotation of a line



In the **Algebra View** type

1. r. A slider for r will be created.
2. A = (r,0,0)(r is the radius of the base of the cylinder. A point A will be plotted.
3. h. A slider for h(the height of the cylinder) will be created.
4. B = (r,0,h). A point B will be plotted.
5. f = Segment(A,B). A line segment AB will be drawn.
6. Type 0deg a slider for angle α will be created.
7. Rotate(f,n,zAxis). A line A'B' coincident with AB will be drawn that will rotate as we move the point on the slider for n.
8. Surface(f,n,zAxis), when you change the value of n on the slider, a cylinder will begin to form, when you move the point move the point on the slider for α to 360°. A cylinder will be formed.
9. You can draw a cylinder with any radius by changing the values of radius and height of the cylinder by moving the point on the slider for r and h.

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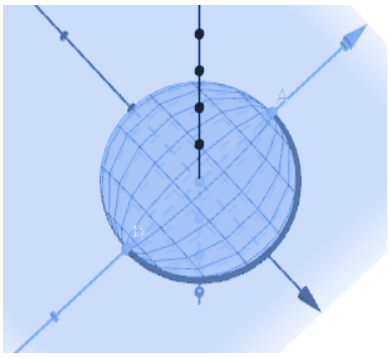
Draw a sphere by rotating a Semicircle



1. Activate 3D Graphic by clicking ☰ on the top and selecting view from the dropdown menu and then selecting 3D Graphic
2. Plot two points A = (a,0) and B = (-a,0).
3. Semicircle(A,B). A semicircle named c will be drawn in the Graphic view.
4. Type 0deg. A slider for α will be created.

5. Rotate(c,α,yAxis). A semicircle A'B' coincident with semicircle AB will be drawn that will rotate if you move the point on the slider for α.
6. Surface(c,α,yAxis). A sphere will be drawn as you move the point on slider from 0° to 360°.

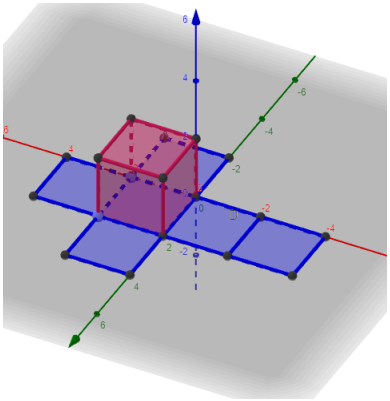
We can rotate any curve about a line to create a surface of revolution by entering the function for f instead of a line segment or semicircle.

Nets of a polyhedron

A net is a pattern that can be folded to form a polyhedron.

We can draw nets of 3D shapes by activating tool and selecting **Net** from its dropdown menu.

Example 1.Draw a cube and its net.



1. Draw a cube by activating tool and selecting **Cube** from its dropdown menu and selecting two points.
2. Draw its net by activating tool and selecting **Net** from its dropdown menu and clicking the cube.
   The net shown below and a slider named b will be shown. As you change the values of b, from 0 to 1, the net will enclose the cube.

Cross-Sections of a Polyhedron by a Plane

The intersection of the plane and the solid is called a **cross-section**. How to draw the cross-sections of a 3D shape, watch the video: https://youtu.be/hEkDIJGAJzc?list=PLIlF_FGjAuZVZTEFth_X9G2Gvt0nFwA9L&t=151 or follow the instructions given below. For the sake of simplicity, We will position the base of a polyhedron on an xy-plane.

1. Draw a polyhedron by activating the tool and selecting a polyhedron. A polyhedron will be drawn.
2. We may like to find a cross-section by a plane parallel to the xy-plane or perpendicular to the xy-plane or a plane forming an angle other than 0 or $\frac{\pi}{2}$.

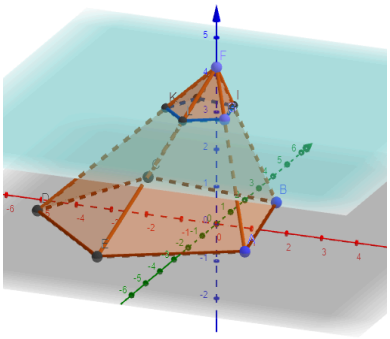
   (i) To draw a plane parallel to xy-plane, activate tool and select **Parallel Plane** from its dropdown menu and then click a point on the z-axis and xy-plane. A plane parallel to the xy plane will be drawn.

   (ii) To draw a plane perpendicular to xy-plane, activate tool and select **Perpendicular Plane** from its dropdown menu and select a line on xy-plane and a point on z-axis. A plane perpendicular to the xy-plane will be drawn.

   (iii) To draw any plane, activate tool and select **Plane through three Points** from its dropdown menu and select a point on each axis. A plane passing through three points will be drawn. If the plane does not intersect the polyhedron, then move one or more points so that it intersects the polyhedron. The cross-section may not be vivid.

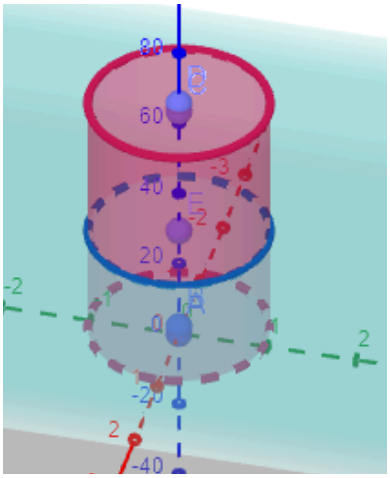
3. To make it vivid, draw its intersection by activating the tool and clicking on a segment of the intersection. It will display a polygon made by the intersection of the polyhedron by the plane.
4. Select the **Move** tool and drag one or more points, different cross-sections will be formed.
5. Hide the axes and xy-plane through the style bar .
6. You can rotate the plane and the 3D shape by activating tool to see it from different angles.

Example 1: Find the cross-sections of a pentagonal pyramid by a plane parallel to the base.

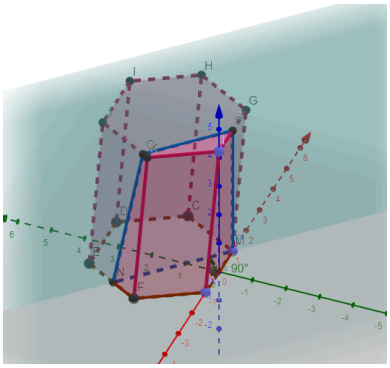


1. Draw a regular pentagon on the xy-plane.
2. Plot a point on z-axis (0,0,a)
3. Draw a regular pentagon on the xOy plane by using the **Polygon** tool.
4. Draw a pentagonal pyramid using the **Pyramid** tool.
5. Select another point on z-axis between origin and (0,0,a)
6. Draw a plane parallel to xOy-plane by activating the tool and selecting **Parallel Plane.** A cross-section of the plane and pentagonal prism will be shown.
7. Activate tool, and select on a segment of the intersection. A pentagon with blue color will be shown as a cross section.

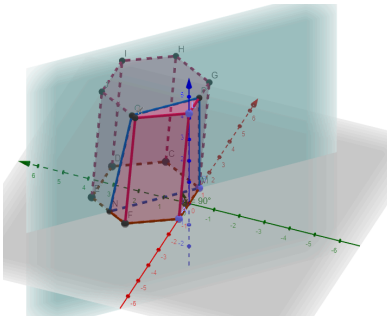
Example 2: Draw the cross-section of a cylinder by a plane parallel to it.
We can draw it in a manner similar to that of a polyhedron by selecting **Cylinder** from the **Polyhedron** tool.



Example 3: Find the cross-sections of a hexagonal prism by a plane perpendicular to the base.



1. Draw a regular pentagon on an xy-plane.
2. Plot a point on z-axis (0,0,a)
3. Draw a regular hexagon on an xy-plane by using the **Polygon** tool.
4. Draw a hexagonal pyramid using the **Pyramid** tool.
5. Select another point on z-axis between origin and (0,0,a)
6. Draw a plane perpendicular to xOy-plane by activating the tool and selecting **Perpendicular Plane.** A cross-section of the plane and pyramid will be shown.
7. Click on a segment of the cross-section,the quadrilateral in blue color will be shown as a cross-section.



Example 4: Find the cross-sections of a cube by a plane:

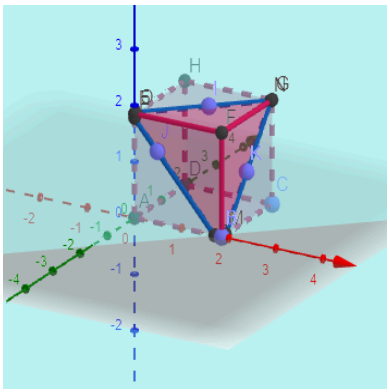
(a) What are the different polygons that you can make for the cross-section? Make them.

(b) How would you position the points to draw a plane so that cross-section will be an equilateral triangle.

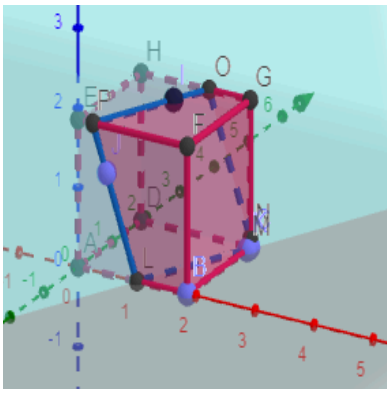
(c) How would you position the points to draw a plane so that cross-section will be a Square.

Answer

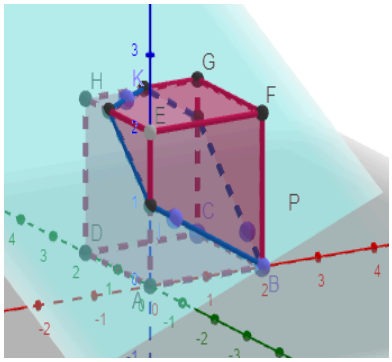
(a) (i) A triangle.



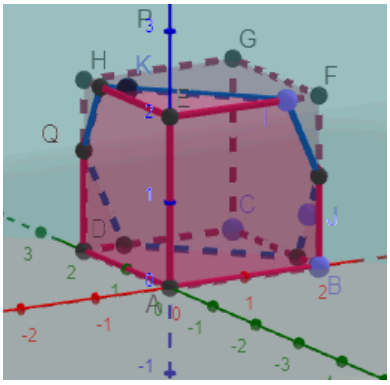
(ii) A quadrilateral



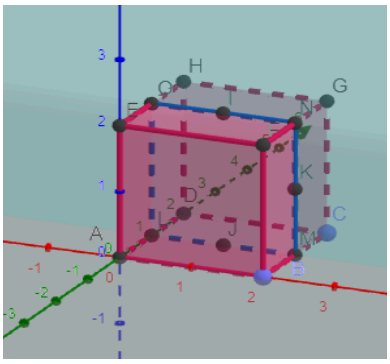
(iii) A pentagon



(iv) A hexagon

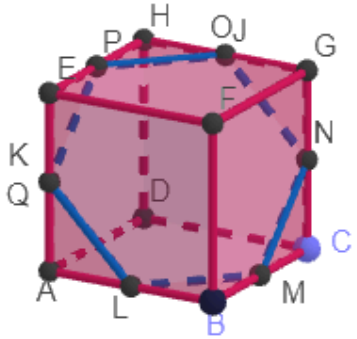


(b) To get a square as a cross-section



1. Draw a cube.
2. Activate tool and select **Parallel or Perpendicular Plane.** It will show a parallel/perpendicular plane intersecting the cube.
3. Activate tool, and click on a segment of the intersection. It will show a square as a cross section.

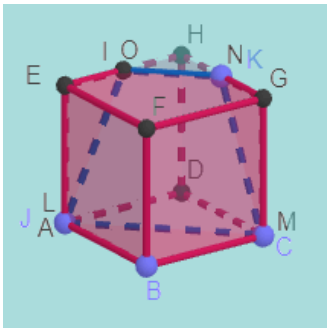
(c) To get a regular hexagon as a cross-section



1. Draw a cube ABCDEFGH.

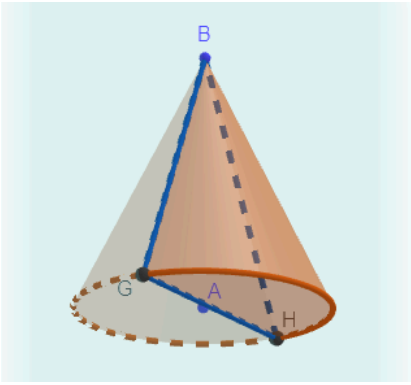
2. Mark midpoints of three edges CG, GH and AE as I, J, K respectively using the Midpoint or Center tool.(Note the edges are along or parallel to x-axis, y-axis, z-axis)
3. Activate tool and select **Plane through three points.** It will show a plane intersecting the cube.
4. Activate tool, and click on a segment of the intersection. It will show a hexagon LMNOPQ as a cross section.

(d) To get an isosceles trapezium as a cross-section



1. Draw a cube ABCDEFGH.
2. Mark midpoints of edges EH and HG as I and K respectively using the Midpoint or Center tool.
3. Activate tool and select **Plane through Three Points** and select points A, C and K. It will show a plane intersecting the cube.
4. Activate tool, and click on a segment of the intersection. It will show an isosceles trapezoid LMNO as a cross section.

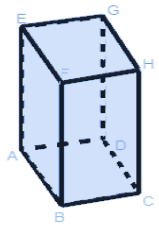
Cross-section of a cone by a plane



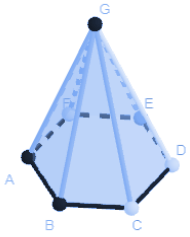
1. Plot points A = (0,0,0) and B = (0,0,a)
2. Draw a cone by activating tool and selecting **Cone** from its dropdown menu, and selecting points A and B and entering r for the radius.
3. Activate tool and select **Plane through three Points** from its dropdown menu and select three points C, D, E on x-axis, y-axis and z-axis respectively. It will show a plane passing through points C, D, and E.
4. Select the **Move** tool and drag one or more of the points C, D, and E points to change the plane.
5. Select tool, and select the plane and the cone and click on a segment of the cross-section. It will show the cross-section(the figure with a boundary in blue color).

Exercise 13: 3D figures and their Properties

1. Name the solids given below:

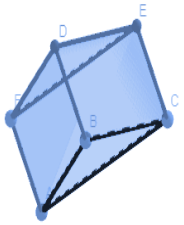


(a)



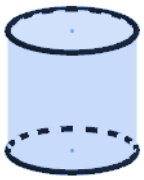
(b)

(c)



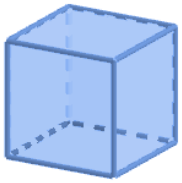
(d)

(e)



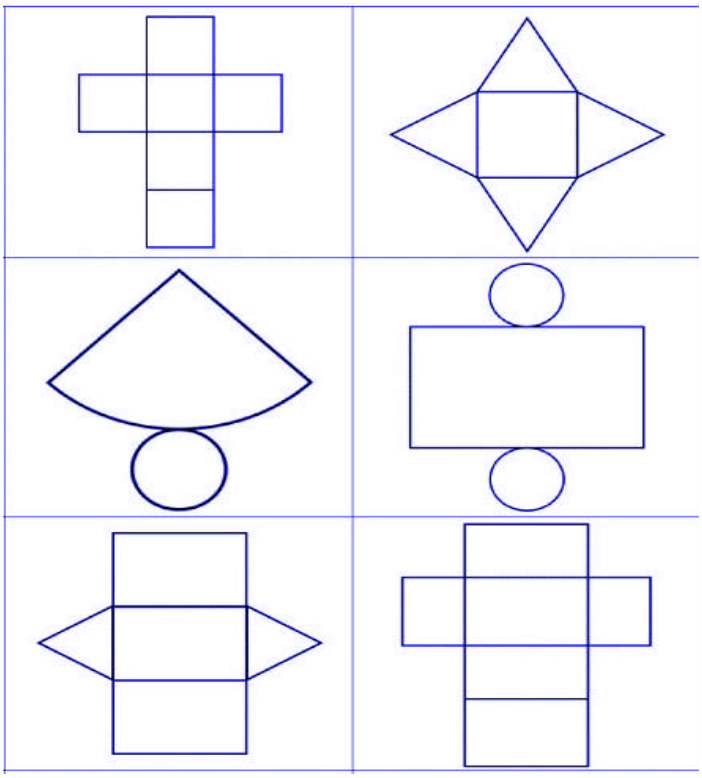
(f)

(g)



(h)

2. For each of the solids in Q.1, state whether the solid is a polyhedron or not. If it is, name the polyhedron and find the number of faces, vertices, and edges.
3. Draw a
   (a) cube
   (b) rectangular prism
   (c) cone
   (d) right circular cylinder
   (e) regular pentagonal prism
   (f) tetrahedron
4. Draw the nets of the following 3-D figures:
   (a) A cube
   (b) A triangular prism
   (c) A hexagonal pyramid
   (d) A rectangular pyramid
5. For each net, name the solid it will make.



6. Find the number of faces, edges, and vertices of the solids given below. What relation do you notice between the number of vertices, edges and faces?

Name of the solid	Number of Faces F	Number of Edges E	Number of Vertices V
Cube			
Triangular Prism			
Pentagonal Prism			
Hexagonal Prism			
A prism whose base is a polygon of n sides			

7. Draw
   (a) A cube whose side is 3 cm.
   (b) A rectangular prism whose length, width, and height are 5 cm, 3 cm and 4 cm respectively.
   (c) A regular pentagonal prism with side 2 cm.
   (d) A tetrahedron whose side is 3 cm.
8. Draw a cylinder the radius of whose base is 1.5 cm and height is 3 cm.

9. Draw a cone whose base has a radius of 1 cm and height is 3 cm.
10. Draw the following 3D shapes and intersect them by a plane parallel to the base and name the shape that you get.
    (a) A rectangular prism
    (b) A cylinder.
    (c) A square pyramid.
    (d) A triangular prism.
    (e) A pentagonal pyramid.
    (f) A Cone
11. Draw the following 3D shapes and intersect them by a plane perpendicular to the base and name the shape that you get.
    (a) A rectangular pyramid.
    (b) A cylinder.
    (c) A cone.
    (d) A triangular prism.
    (e) A pentagonal pyramid.
    (f) A square pyramid.
12. Draw a cube and a plane intersecting a cube to get the following cross-sections.
    (a) A scalene triangle
    (b) An isosceles triangle.
    (c) An equilateral triangle.
    (d) a trapezoid.
    (e) An isosceles triangle.
    (f) A parallelogram.
    (g) A rhombus.
    (h) A pentagon
    (i) A hexagon.
13. Demonstrate that the length of the diagonal of a cube with a side equal to a cm is $\sqrt{3}$a. For the concept of diagonal of a cube, watch the video: https://youtu.be/kajYB_KlKeE?list=PLlIF_FGjAuZV_GYPFthdZ77StiFM_PSQw&t=6
14. Find the length of diagonals of a $5\times4\times6$ rectangular prism.

14. Surface Area and Volume of 3D Figures

For concept of surface area and volume, watch the video: https://youtu.be/qJwecTgce6c?t=7
Formulas for surface area and volume of 3D shapes watch the video: https://youtu.be/eBAq_caikJ4?t=553
Why the volume of a square pyramid is $\frac{1}{3}$ the volume of the cube with the same base and height equal to the length of the side of a cube, watch the video: https://youtu.be/l-3Gl8emteU?t=36

Verifying the volume and surface areas of polyhedrons

1. Draw a polyhedron with given dimensions.
2. Its volume will be given in the **Algebra view**.
3. To find its surface area, draw its net. The surface area of the net would be shown in the **Algebra view.** You can find the area of each face by activating the **Area** tool and clicking on the face.

Understanding the Formulae for surface areas of cylinder, cone and sphere

We can find the surface areas of these by finding their nets. Watch the videos given below to see their nets.
Sphere: https://youtu.be/6g4TW8wnQ_I?t=51
Cylinder: https://www.youtube.com/shorts/qMw4nN6BIIs?feature=share
Cone: https://youtu.be/KO9K8q8j8ow?t=397

Verifying the Volume for cylinder, cone and sphere

1. Draw the figure with given dimensions.
2. Measure the volume of the solid by activating tool and selecting **Volume.** It will give volume of the solid. It is also given in the **Algebra View**.

Exercise 14: Surface Area and Volume of 3D Figures

1. Plot the points (1,2,3), (0,0,0), (2,0,0) and (0,0,2)
2. Find the distance between points (2, 0, 3), and (5, 4, 2).
3. Find the coordinates of the midpoint of the line segment whose endpoints are (1,4, 6) and (3,2,4).
4. (a) Draw a cube whose side is 3 cm.
   (b) Find its volume.
   (c) Find its surface area.
5. (a) Draw a rectangular prism with length 5 cm width 3 cm and height is 4 cm.
   (b) Find its volume.
   (c) Find its surface area.
6. (a) Draw a cylinder whose base has a radius of 3 cm and a height of 4 cm.
   (b) Find its volume.
   (c) Find its surface area.
7. (a) Draw a cone whose base has a radius of 4 cm and a height of 5 cm.
   (b) Find its volume.
   (c) Find its surface area.
8. (a) Draw a sphere that has a radius of 3 cm.
   (b) Find its volume.
   (c) Find its surface area.
9. A regular hexagonal prism is 6 cm high and has a base area of 24 cm^2 , find its
   (a) volume.
   (b) surface area
10. A regular pentagonal pyramid is 6 cm high and its base has an area of 20 cm^2, find
    (a) its volume.
    (b) its surface area.
11. The volume of a square prism is 108 cm^3 and its height is 3 cm. Find
    (a) the length of the side of the square
    (b) the surface area of the prism.
12. The volume of a cylinder is 324 cm^3 and its height is 4 cm. Find its
    (a) radius.
    (b) surface area.
    (c) Can you draw another cylinder with these dimensions? If yes, draw it.

Links for More on GeoGebra

1. **GeoGebra Website**: http://www.geogebra.org
2. **GeoGebra Forum**: www.geogeba.org/forum/
3. **GeoGebra on Facebook**: www.facebook.com/geogebra
4. **GeoGebra's YouTube Channel**: http://www.youtube.com/user/GeoGebraChannel
5. **GeoGebraTube**: http://www.geogebratube.org
6. Gerry Stahl **Dynamic-Geometry Activities with GeoGebra for Virtual Math Teams** https://gerrystahl.net/vmt/activities.pdf
7. Phelps, Steve **Introduction to GeoGebra:** https://fdocuments.in/document/an-introduction-to-geogebra.html?page=1#google_vignette
8. **Tutorials for beginning to advanced users**: http://math4allages.wordpress.com/geogebra/